Al-Farabi Kazakh National University
and
Ghent University

Department of Fundamental Mathematics
Faculty of Mechanics and Mathematics

Department of Mathematics:
Analysis, Logic and Discrete
Mathematics
Faculty of Sciences

Aidyn Kassymov

Basic functional and geometric inequalities for the fractional order operators on homogenous Lie groups

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Prof. Dr. Michael Ruzhansky
Assoc. Prof. Dr. Durvudkhan Suragan

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Preface

This dissertation is devoted to study of the fractional functional and geometric inequalities on homogeneous Lie groups. More precisely, we develop the fractional calculus and non-commutative analysis, i.e., we combined two big direction in mathematics. This perspective turned out to be extremely useful on both a conceptual and a technical level. Namely, we will systematically employ the ideas of Prof. Michael Ruzhansky, Assoc. Prof. Durvudkhan Suragan, Assoc. Prof. Berikbol Torebek, Assoc. Prof. Niyaz Tokmagambetov, Dr. Nurgissa Yessirkegenov, Dr. Bolys Sabitbek and others.

In Chapter 2, we give main definitions and preliminary results from [1], [2] and open access books [3] and [4], which both received the "Ferran Sunyer i Balauguer Award" in 2016 and 2019, respectively. Also, we give definition of the fractional Sobolev space on homogeneous Lie groups and integer order of the Sobolev space on graded, stratified Lie groups.

In Chapter 3, we develop theory of the fractional functional and geometric inequalities on homogeneous Lie groups. We obtain the fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic fractional inequalities which is even new on Euclidean case. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which means boundedness of the Riesz operator in $L^q - L^p$ spaces. Also, we obtain the Stein-Weiss inequality (or a radially weighted Hardy-Littlewood-Sobolev inequality) for the Riesz potential. In addition, we show integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups. This chapter is based on the papers [5], [6], [7] (joint works with M. Ruzhansky and D. Suragan), [8], [9], [10] (joint works with D. Suragan) and [11] (joint work with A. Kashkynbayev and D. Suragan).

In Chapter 4, we study a question of the reverse functional inequalities. Firstly, we start to study reverse integral Hardy inequality on metric measure space. We note that, in the work [12], authors introduced polar decomposition on metric measure space, which is play a key role in their proof. In this chapter, we obtain reverse integral Hardy inequality on metric measure space with parameters $q < 0$ and $p \in (0, 1)$. As consequences, we get integral reverse Hardy inequality on on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifoldse with parameters $q < 0$ and $p \in (0, 1)$. Also, we show integral reverse Hardy inequality on metric measure space with parameters $\infty < q \leq p < 0$ and as a consequences we show reverse integral Hardy inequality on homogeneous Lie groups. Then we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$ and $p \in (0, 1)$. Also, we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss type and improved Stein-Weiss type inequalities with parameters $\infty < q \leq p < 0$, which is even new in Euclidean settings. In addition, we obtain the reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups. This chapter is based on the papers [13], [14] (joint works with M. Ruzhansky and D. Suragan), [15] (joint work with D. Suragan) and [16].

In Chapter 5, we give applications of the functional inequalities in PDE. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. As a application of Lyapunov's inequality, we give lower estimate of the first eigenvalue of the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show existence of the weak solution for the nonlinear equation with the p-sub-Laplacian on the Heisenberg and stratified groups. Also, we show existence of the weak solution for the nonlinear equation with the fractional sub-Laplacian and Hardy potential on homogeneous Lie groups and multiplicity of the weak solution with first stratum Hardy potential on Heisenberg and stratified groups. Then we discuss blow-up results for heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous Lie groups and for heat equation with sub-Laplacian and logarithmic nonlinearity on stratified group. Also, we show blow-up results for viscoelastic equations with sub-Laplacian on stratified groups, heat and wave Rockland equations on graded groups. This chapter is based on the papers [5], [7] (joint works with M. Ruzhansky and D. Suragan), [8], [9], [17], [18] (joint works with D. Suragan), [11], [19] (joint works with A. Kashkynbayev and D. Suragan), [20] (joint work with B. Torebek and N. Tokmagambetov), [21] (joint work with B. Bekbolat and N. Tokmagambetov) and [22].

In Appendix, we consider one-dimensional functional inequalities on Euclidean case. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of this inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem. Appendix is based on the papers [23] (joint work with M. Ruzhansky, B. Torebek and N. Tokmagambetov) and [24] (joint work with M. Kirane and B. Torebek).

Almaty, Ghent, April 2020 — Aidyn Kassymov

Summary

In this PhD dissertation, we study functional and geometric inequalities on homogeneous Lie groups. For the direct inequalities we obtain fractional Hardy, Sobolev, Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, logarithmic inequalities, Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. Also, we obtain integer order Sobolev-Folland-Stein inequality on stratified groups.

For the reverse inequalities, we prove reverse integral Hardy inequalities with parameters $q < 0$, $p \in (0, 1)$ and $-\infty < q \leq p < 0$. Also, we show reverse integral Hardy inequalities on homogeneous Lie groups, hypebolic space and Cartan-Hadamard manifolds with $q < 0$, $p \in (0, 1)$. As a consequences, we show reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved version Stein-Weiss inequalities for the cases $q < 0$, $p \in (0, 1)$ and $-\infty < q \leq p < 0$. In addition, we obtained the reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

Then we show some applications of these inequalities in linear and nonlinear PDE on homogeneous groups.

Also, we consider one-dimensional functional inequalities on Euclidean case. We establish fractional Hardy, Poincaré type, Gagliardo-Nirenberg type and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of these inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.

Samenvatting

In dit proefschrift bestuderen we functionele en geometrische ongelijkheden bij homogene Lie-groepen. Voor de directe ongelijkheden verkrijgen we fractionele Hardy, Sobolev, Hardy-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, logaritmische ongelijkheden, Hardy-Littlewood-Sobolev en Stein-Weiss ongelijkheden op homogene Lie-groepen. We verkrijgen ook een geheel aantal Sobolev-Folland-Stein-ongelijkheid voor gelaagde groepen.

Voor de omgekeerde ongelijkheden, bewijzen we omgekeerde integrale Hardy ongelijkheden met parameters $q < 0,\ p \in (0, 1)$ en $-\infty < q \leq p < 0$. We tonen ook omgekeerde integrale Hardy-ongelijkheden op homogene Lie-groepen, hyperbolische ruimte en Cartan-Hadamard-spruitstukken met $q < 0,\ p \in (0, 1)$. Als gevolg hiervan tonen we omgekeerde Hardy-Littlewood-Sobolev, Stein-Weiss en verbeterde versie Stein-Weiss ongelijkheden voor de gevallen $q < 0,\ p \in (0, 1)$ en $-\infty < q \leq p < 0$. Bovendien verkrijgen we de omgekeerde Hardy, L^p - Sobolev en L^p - Caffarelli-Kohn-Nirenberg ongelijkheden met de radiale derivaat op homogene Lie-groepen.

Vervolgens tonen we enkele toepassingen van deze ongelijkheden in lineaire en niet-lineaire PDE op homogene groepen.

We hebben ook rekening gehouden met eendimensionale functionele ongelijkheden in Euclidisch geval. We hebben fractionele Hardy, Poincaré type, Gagliardo-Nirenberg en Caffarelli-Kohn-Nirenberg ongelijkheden vastgesteld voor de fractionele orde differentiële operatoren als Caputo, Riemann-Liouville en Hadamard fractionele derivaten. Ook tonen we toepassingen van deze ongelijkheden. Daarnaast tonen we Lyapunov en Hartman-Wintner-type ongelijkheden voor een fractionele partiële differentiaalvergelijking met Dirichlet-voorwaarde, geven we een toepassing van deze ongelijkheden voor de eerste eigenwaarde en tonen we de La Vallé Poussin-type ongelijkheid voor probleem met fractionele elliptische grenswaarden.

Contents

1. Introduction

The first mathematicians who study of subelliptic analysis on the Heisenberg group were Folland and Stein in [25], who consistently created a generalisation of the analysis for more general stratified groups [26]. And it can also be noted that Rothschild and Stein generalised these results for general vector fields satisfying the Hörmander's condition. We can say that these results were published in the famous book by Folland and Stein [1] which laid the anisotropic analysis. And it is worth noting that homogeneous Lie group is nilpotent.

The history of fractional calculus originates from the works of Riemann and Liouville. And in these works, the concepts of the fractional integral were introduced for the first time. Along with integer derivatives, the concept of a fractional derivative was introduced, which was named after Riemann and Liouville. Then, Hadamard in his works, he introduced a different definition of the fractional derivative. And it is also worth noting that Caputo also introduced the definition of a fractional derivative that in a particular case can be equal to the Riemann-Liouville derivative. These operators are non-local operators. Note that these fractional derivatives are one-dimensional operators. For the multidimensional case, the concept of a multidimensional fractional Laplacian is introduced via Laplacian's symbol. It is worth noting that fractional calculus is currently a rapidly developing mathematical field. The main aim of this dissertation is to combine non-commutative analysis on groups and fractional calculus.

Nowadays, functional and geometric inequalities on Lie groups are currently a rapidly developing field of mathematics. Many nonlinear differential equations of problems of mechanics and problems of physics to which the global solvability of problems is proved through functional inequalities. It means, one of the most important tool to study PDE is the functional inequalities. For example, integer order multi-dimensional Hardy inequality demonstrates the following inequality:

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \quad 1 < p < n, \ \ \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

where $|\cdot|$ is the Euclidean distance and constant $\left(\frac{p}{n-p}\right)^p$ is a sharp. This inequality has applications in a lot of areas of mathematics, for example in spectral theory. Also, by this inequality we can show Heisenberg-Pauli uncertainly principle, which has application in quantum theory. Firstly, on group settings Hardy inequality was obtained by Garofalo and Lanconelli on Heisenberg group in [27]. On stratified groups, Hardy inequality were obtained in the works [28], [29] and [30], on homogeneous groups was obtianed in [31] and on graded groups in [32]. In [33] the authors studied the fractional p-Laplacian and established the following fractional L^p-Hardy inequality

$$C \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy, \tag{1.2}$$

where $u \in C_0^\infty(\mathbb{R}^N)$ and $C > 0$ is a positive constant. Also, best constant was obtained in [33]. Generelasation of this inequality was obtained in [34].

Classical Sobolev inequality (or a continuous Sobolev embedding) is the one of the most popular functional inequality. Sobolev inequality has a lot of applications in

PDE and variational principles. Let $\Omega \subset \mathbb{R}^N$ be a measurable set and $1 < p < N$, then the (classical) Sobolev inequality is formulated as

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \;\; u \in C_0^\infty(\Omega), \tag{1.3}$$

where $C = C(N,p) > 0$ is a positive constant, $p^* = \frac{Np}{N-p}$ and ∇ is a standard gradient in $\mathbb{R}^N$ (see e.g., [35]). Logarithmic Sobolev inequality was proverd in [36] and it has the following form:

$$\int_{\mathbb{R}^N} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right) dx \leq \frac{N}{p}\log\left(C\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^p(\mathbb{R}^N)}}\right), \; 1 \leq p < \infty, \tag{1.4}$$

where $u, \nabla u \in L^p(\mathbb{R}^N)$. On Heisenberg groups case Sobolev inequality inequality was obtained by Folland and Stein, on stratified groups by Garofalo and Vassilev in [37], on graded groups by Fischer and Ruzhansky in [3]. Also, the best constant of the Sobolev inequality for general hypoelliptic (Rockland operators) on general graded Lie groups was obtained in [38]. For the fractional order Sobolev's inequality was obtained in [39] in the case $N > sp$, $1 < p < \infty$, and $s \in (0,1)$, for any measurable and compactly supported function u one has

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C[u]_{s,p}, \tag{1.5}$$

where $C = C(N,p,s) > 0$ is a suitable constant, $[u]^p_{s,p} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$ and $p^* = \frac{Np}{N-sp}$. There is a number of generalisations and extensions of above Sobolev's inequality. For example, in [34] the authors proved the following weighted fractional Sobolev inequality: Let $1 < p < \frac{N}{s}$ and $0 < \beta < \frac{N-ps}{2}$, then for all $u \in C_0^\infty(\mathbb{R}^N)$ one has

$$C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}|x|^\beta|y|^\beta}dxdy \geq \left(\int_{\mathbb{R}^N}\frac{|u|^{p^*}}{|x|^{\frac{2\beta p^*}{p}}}dx\right)^{\frac{p}{p^*}}, \tag{1.6}$$

where $C = C(N,p,s) > 0$ and $p^* = \frac{Np}{N-sp}$.

E. Gagliardo and L. Nirenberg independently, obtained following (interpolation) inequality

$$\|u\|^p_{L^p(\mathbb{R}^N)} \leq C\|\nabla u\|^{N(p-2)/2}_{L^2(\mathbb{R}^N)}\|u\|^{(2p-N(p-2))/2}_{L^2(\mathbb{R}^N)}, \;\; u \in H^1(\mathbb{R}^N), \tag{1.7}$$

where

$$\begin{cases} 2 \leq p \leq \infty \text{ for } N = 2, \\ 2 \leq p \leq \frac{2N}{N-2} \text{ for } N > 2. \end{cases}$$

In particular case, from this inequality we can obtain Sobolev inequality. In addition, the logarithmic Gagliardo-Nirenberg inequality was proved in [36] and its fractional version was proved in [40]. On Heisenberg group, the Gagliardo-Nirenberg inequality has the following form

$$\|u\|^p_{L^p(\mathbb{H}^n)} \leq C\|\nabla_{\mathbb{H}^n} u\|^{Q(p-2)/2}_{L^2(\mathbb{H}^n)}\|u\|^{(2p-Q(p-2))/2}_{L^2(\mathbb{H}^n)}, \tag{1.8}$$

where $\nabla_{\mathbb{H}}$ is a horizontal gradient and Q is a homogeneous dimension of $\mathbb{H}^n$. Also, in [38] authors obtained Gagliardo-Nirenberg inequality and its the best constants on general hypoelliptic (Rockland operators) on general graded Lie groups. Fractional version of the Gagliardo-Nirenberg was established in [41]:

$$\|u\|_{L^{\tau}(\mathbb{R}^N)} \leq C[u]^{a}_{s,p}\|u\|^{1-a}_{L^{\alpha}(\mathbb{R}^N)}, \ \forall u \in C^1_c(\mathbb{R}^N), \tag{1.9}$$

for $N \geq 1,\ s \in (0,1),\ p > 1,\ \alpha \geq 1,\ \tau > 0$, and $a \in (0,1]$ is such that

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{N}\right) + \frac{1-a}{\alpha}.$$

In the fundamental work of the L. Caffarelli, R. Kohn and L. Nirenberg in [42], they obtained:

Theorem 1.1. *Let $N \geq 1$, and let $l_1,\ l_2,\ l_3,\ a,\ b,\ d,\ \delta \in \mathbb{R}$ be such that $l_1, l_2 \geq 1$, $l_3 > 0,\ 0 \leq \delta \leq 1$, and*

$$\frac{1}{l_1} + \frac{a}{N},\ \frac{1}{l_2} + \frac{b}{N},\ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0. \tag{1.10}$$

Then,

$$\||x|^{\delta d+(1-\delta)b}u\|_{L^{l_3}(\mathbb{R}^N)} \leq C\||x|^{a}\nabla u\|^{\delta}_{L^{l_1}(\mathbb{R}^N)}\||x|^{b}u\|^{1-\delta}_{L^{l_2}(\mathbb{R}^N)},\ \ u \in C^{\infty}_c(\mathbb{R}^N), \tag{1.11}$$

if and only if

$$\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta\left(\frac{1}{l_1} + \frac{a-1}{N}\right) + (1-\delta)\left(\frac{1}{l_2} + \frac{b}{N}\right),$$
$$a - d \geq 0, \quad \text{if } \delta > 0,$$
$$a - d \leq 1, \quad \text{if } \delta > 0 \ \text{ and } \ \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a-1}{N}, \tag{1.12}$$

where C is a positive constant independent of u.

The logarithmic analogue of the Caffarelli-Kohn-Nirenberg inequality was proved in [43]. Recently many different versions of Caffarelli-Kohn-Nirenberg inequalities have been obtained, namely, in [44] on the Heisenberg groups, in [45] and [29] on stratified groups, in [46] on (general) homogeneous Lie groups. In [41] the authors proved the fractional analogues of the Caffarelli-Kohn-Nirenberg inequality in weighted fractional Sobolev spaces. Also, a fractional Caffarelli-Kohn-Nirenberg inequality for an admissible weight in $\mathbb{R}^N$ was obtained in [34].

One of the pioneering work of Hardy and Littlewood in [47], they considered the 1D fractional integral operator on $(0,\infty)$ given by

$$T_{\lambda}u(x) = \int_0^{\infty} \frac{u(y)}{|x-y|^{\lambda}}dy, \quad 0 < \lambda < 1, \tag{1.13}$$

and proved the following theorem:

Theorem 1.2. *Let $1 < p < q < \infty$ and $u \in L^p(0,\infty)$ with $\frac{1}{q} = \frac{1}{p} + \lambda - 1$, then*

$$\|T_{\lambda}u\|_{L^q(0,\infty)} \leq C\|u\|_{L^p(0,\infty)}, \tag{1.14}$$

where C is a positive constant independent of u.

The multi-dimensional analogue of (1.13) can be written by the formula:

$$I_\lambda u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < N. \tag{1.15}$$

The multi-dimensional case of Theorem 1.2 was extended by Sobolev in [48]:

Theorem 1.3. *Let $1 < p < q < \infty$, $u \in L^p(\mathbb{R}^N)$ with $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$, then*

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \le C\|u\|_{L^p(\mathbb{R}^N)}, \tag{1.16}$$

where C is a positive constant independent of u.

Then, in [49] Stein and Weiss obtained the radially weghted extention of the Hardy-Littlewood-Sobolev inequality, which is known as the Stein-Weiss inequality.

Theorem 1.4. *Let $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \ge 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{N} - 1$. If $1 < p \le q < \infty$, then*

$$\||x|^{-\beta} I_\lambda u\|_{L^q(\mathbb{R}^N)} \le C\||x|^\alpha u\|_{L^p(\mathbb{R}^N)}, \tag{1.17}$$

where C is a positive constant independent of u.

On the Heisenberg group, the Hardy-Littlewood-Sobolev inequality was proved by Folland and Stein in [25] and an analogue of Stein-Weiss inequality was proved in [50]. In [51] the authors studied the Stein-Weiss inequality on the Carnot groups. We also note that the best constant in the Hardy-Littlewood-Sobolev inequality on the Heisenberg group is now known (see Frank and Lieb [52]) and in the Euclidean case this was done earlier by Lieb in [53].

The reverse Stein-Weiss inequality in Euclidean setting has the following form:

Theorem 1.5 ([54], Theorem 1)**.** *For $n \ge 1, p \in (0,1), q < 0, \lambda > 0, 0 \le \alpha < -\frac{n}{q}$, and $0 \le \beta < -\frac{n}{p'}$ satisfying $\frac{1}{p} + \frac{1}{q'} - \frac{\alpha+\beta+\lambda}{n} = 2$, there is a constant $C = C(n, \alpha, \beta, \lambda, p, q) > 0$ such that for any non-negative functions $f \in L^{q'}(\mathbb{R}^n)$ and $0 < \int_{\mathbb{R}^n} g^p(y)dy < \infty$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x|^\alpha |x-y|^\lambda f(x)g(y)|y|^\beta dydx \ge C\left(\int_{\mathbb{R}^n} f^{q'}(x)dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{\frac{1}{p}}, \tag{1.18}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

The inequality (1.18) is equivalent to,

$$\left(\int_{\mathbb{R}^n} |x|^{\alpha q} \left(\int_{\mathbb{R}^n} |x-y|^\lambda |y|^\beta g(y)dy\right)^q dx\right)^{\frac{1}{q}} \ge C\left(\int_{\mathbb{R}^n} g^p(y)dy\right)^{\frac{1}{p}}. \tag{1.19}$$

From last, if $\alpha = \beta = 0$ we obtain the reverse Hardy-Littlewood-Sobolev inequality. Improved Stein-Weiss inequality was obtained in [55] on Euclidean upper half-space. For more results about the reverse Hardy–Littlewood–Sobolev inequality in Euclidean space, we refer the reader to [56] [57], [58], [59] and the references therein.

By summarising above facts, in this dissertation we developed direct and reverse inequalities on homogeneous groups. In Chapter 3, we obtain fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic fractional inequalities. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which means boundedness of the Riesz operator in $L^q - L^p$ spaces. Also, we obtain the Stein-Weiss inequality for the Riesz potential. In addition, we show integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups.

In Chapter 4, we prove reverse integral Hardy inequality on metric measure space with $q < 0$ and $p \in (0,1)$ and $\infty < q \leq p < 0$, integral reverse Hardy inequality on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. As consequences we show Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$, $p \in (0,1)$ and $\infty < q \leq p < 0$. In addition, we obtain the reverse Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

In Chapter 5, we give applications of the functional inequalities to PDE. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show the existence of the weak solution for the nonlinear equation with the p-sub-Laplacian on the Heisenberg and stratified groups and we show the existence of the weak solution for the nonlinear equation with the fractional sub-Laplacian and Hardy potential on homogeneous Lie groups. Then we discuss blow-up results for heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous Lie groups, for heat equation with sub-Laplacian and logarithmic nonlinearity on stratified groups, viscoelastic equation on stratified groups, heat and wave Rockland equations on graded groups. We give introduction in every section of this chapter.

In Appendix, we consider one-dimensional functional inequalities on Euclidean case. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of this inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.

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- A. Kassymov, M. Ruzhansky and D. Suragan. Fractional logarithmic inequalities and blow-up results with logarithmic nonlinearity on homogeneous groups. *Nonlinear Differ. Equ. Appl.*, 27:7, 2020. (Scopus, Web of Science, Q1);
- A. Kassymov, M. Ruzhansky and D. Suragan. Hardy-Littlewood-Sobolev and Stein-Weiss inequalities on homogeneous Lie groups. *Integral Transform. Spec. Funct.,* 30(8):643–655, 2019. (Scopus, Web of Science, Q2);

- A. Kassymov and D. Suragan. Existence of solutions for p-sub-Laplacians with nonlinear sources on the Heisenberg group. *Complex Variables and Elliptic Equations*, dx.doi:10.1080/17476933.2020.1731737, 2020. (Scopus, Web of Science, Q2);
- A. Kassymov, B. Torebek and N. Tokmagambetov. Nonexistence Results for the Hyperbolic-Type Equations on Graded Lie Groups. *Bulletin of the Malaysian Mathematical Sciences Society*, doi:10.1007/s40840-020-00919-6, 2020, (Scopus, Web of Science, Q2);
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2. Preliminaries

In this chapter, we briefly give definitions, main properties and theorems of the homogeneous, graded, stratified Lie groups and Heisenberg groups. Also, we will fix the main notations in this dissertation. All main definitions were taken from [1], [2] and open access books [3] and [4].

2.1. Homogeneous Lie groups.

In whole of this dissertations, any Lie algebra $\mathfrak{g}$ is assumed to be real and finite dimensional. The lower central series of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \quad \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}],$$

terminates at 0 in a finite number of steps. If the lower central series of the Lie algebra $\mathfrak{g}$ terminates at 0 in a finite number of steps, then this Lie algebra is called nilpotent. Then, if $\mathfrak{g}_{(s+1)} = \{0\}$ and $\mathfrak{g}_{(s)} \neq \{0\}$, then $\mathfrak{g}$ is said to be nilpotent of step s. A Lie groups $\mathbb{G}$ is nilpotent (of step s) whenever its Lie algebra is nilpotent (of step s). If $\exp : \mathfrak{g} \to \mathbb{G}$ is the exponential map, by the Campbell-Hausdorff formula for $X, Y \in \mathbb{G}$ sufficiently close to 0 we have

$$\exp X \exp Y = \exp H(X, Y), \tag{2.1}$$

where $H(X, Y)$ is the Campbell-Hausdorff series which is an infinite linear combination of X and X and their iterated commutators and H is universal, i.e. independent of $\mathfrak{g}$, and that

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \dots, \tag{2.2}$$

where the dots indicate terms of order ≥ 3. If $\mathfrak{g}$ is nilpotent, the Campbell-Hausdorff series terminates after finitely many terms and defines a polynomial map from $V \times V$ to V, where V is the underlying vector space of $\mathfrak{g}$. Let us give the following property about Haar measure (see e.g., [3] and [4]).

Proposition 2.1 ([4, Proposition 1.1.1], [3, Proposition 1.6.6] and [1, Proposition 1.2])**.** *Let $\mathbb{G}$ be a connected and simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then if μ denotes a Lebesgue measure on $\mathfrak{g}$, then $\mu \circ \exp^{-1}$ is a bi-invariant Haar measure on $\mathbb{G}$.*

From [3] and [4], a family of dilations of a Lie algebra $\mathfrak{g}$ is a family of linear mappings of the form

$$D_\lambda = \mathrm{Exp}(A \ln \lambda) = \sum_{k=0}^{\infty} (\ln(\lambda) A)^k, \tag{2.3}$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, and D_λ is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects to the Lie bracket:

$$\forall X, Y \in \mathfrak{g}, \ \lambda > 0, \ [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y]. \tag{2.4}$$

Let us give definition of the homogeneous Lie groups, (see e.g., [4, Definition 1.1.6] and [3, Definition 3.1.7]):

Definition 2.2 (Homogeneous Lie group)**.** A *homogeneous (Lie) group* is a connected simply connected Lie group whose Lie algebra is equipped with dilations.

Also, we have another definition of homogeneous Lie group (see [2]):

Definition 2.3 (Homogeneous Lie group)**.** A Lie group (on $\mathbb{R}^N$) $\mathbb{G}$ with the dilation

$$D_\lambda(x) := (\lambda^{\nu_1}x_1, \ldots, \lambda^{\nu_N}x_N),\ \nu_1, \ldots, \nu_n > 0,\ D_\lambda : \mathbb{R}^N \to \mathbb{R}^N,$$

which is an automorphism of the group $\mathbb{G}$ for each $\lambda > 0$, is called a *homogeneous (Lie) group.*

For simplicity, in this dissertation we use the notation λx for the dilation D_λ. We denote

$$Q := \nu_1 + \ldots + \nu_N, \tag{2.5}$$

the homogeneous dimension of a homogeneous group $\mathbb{G}$. Let dx denote the Haar measure on $\mathbb{G}$ and let $|S|$ denote the corresponding volume of a measurable set $S \subset \mathbb{G}$. Then we have

$$|D_\lambda(S)| = \lambda^Q|S| \quad \text{and} \quad \int_{\mathbb{G}} f(\lambda x)dx = \lambda^{-Q}\int_{\mathbb{G}} f(x)dx. \tag{2.6}$$

Then we have the following widely use proposition in our dissertation.

Proposition 2.4 ([4, p. 19])**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension Q, $r > 0$ and dx be a Haar measure. Then, we have*

$$d(rx) = r^Q dx.$$

Definition 2.5 ([4, Definition 1.2.1])**.** For any homogeneous group $\mathbb{G}$ there exists homogeneous quasi-norm, which is a continuous non-negative function

$$\mathbb{G} \ni x \mapsto |x| \in [0, \infty), \tag{2.7}$$

with the properties

a) $|x| = |x^{-1}|$ for all $x \in \mathbb{G}$,
b) $|\lambda x| = \lambda|x|$ for all $x \in \mathbb{G}$ and $\lambda > 0$,
c) $|x| = 0$ iff $x = 0$.

Let us define quasi-ball centered at x with radius r in the following form:

$$B(x, r) := \{x \in \mathbb{G} : |x^{-1}y| < r\}. \tag{2.8}$$

Then we have the following proposition about triangle inequality of the quasi-norm, which is widely use in our proofs.

Proposition 2.6 ([4, Proposition 1.2.4])**.** *Let $\mathbb{G}$ be a homogeneous Lie group. Then there exists a homogeneous quasi-norm on $\mathbb{G}$ which is a norm, that is, a homogeneous quasi-norm $|\cdot|$ which satisfies the triangle inequality*

$$|xy| \le |x| + |y|, \ \ \forall x, y \in \mathbb{G}. \tag{2.9}$$

Furthermore, all homogeneous quasi-norms on $\mathbb{G}$ are equivalent.

Also, let us also recall a well-known fact about quasi-norms.

Proposition 2.7 ([3], Proposition 3.1.38 and [4], Proposition 1.2.4)**.** *If $|\cdot|$ is a homogeneous quasi-norm on $\mathbb{G}$, there exists $C>0$ such that for every $x,y\in\mathbb{G}$, we have*

$$|xy|\leq C(|x|+|y|). \tag{2.10}$$

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs.

Proposition 2.8 ([4, Proposition 1.2.10] and [3, Proposition 3.1.42])**.** *Let $\mathbb{G}$ be a homogeneous Lie group and $\mathfrak{S}:=\{x\in\mathbb{G}:\,|x|=1\}$, be the unit sphere with respect to the homogeneous quasi-norm $|\cdot|$. Then there is a unique Radon measure σ on $\mathfrak{S}$ such that for all $f\in L^1(\mathbb{G})$, we have*

$$\int_{\mathbb{G}} f(x)dx=\int_0^{\infty}\int_{\mathfrak{S}} f(ry)r^{Q-1}d\sigma(y)dr. \tag{2.11}$$

Let us give main definitions of the fractional Sobolev space on homogeneous Lie groups. Assume that $p\geq 1$ and for any measurable function $u:\mathbb{G}\to\mathbb{R}$ we define following quasi-seminorm which is called the Gagliardo quasi-seminorm in the following form

$$[u]_{s,p}:=\left(\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{1}{p}},\quad s\in(0,1),\ Q>1, \tag{2.12}$$

where $|\cdot|$ is a quasi-norm which is defined in Definition 2.5. By $W^{s,p}(\mathbb{G})$ we call the fractional Sobolev spaces on homogeneous groups. For $p\geq 1$ and $s\in(0,1)$, the functional space

$$W^{s,p}(\mathbb{G})=\{u:u\in L^p(\mathbb{G}),[u]_{s,p}<+\infty\}, \tag{2.13}$$

is called the fractional Sobolev space on $\mathbb{G}$.

If $\Omega\subset\mathbb{G}$ is a Haar measurable set, we define the Sobolev space

$$W^{s,p}(\Omega)=\{u:u\in L^p(\Omega),[u]_{s,p,\Omega}=\left(\int_{\Omega}\int_{\Omega}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{1}{p}}<+\infty\}. \tag{2.14}$$

Let us define $W_0^{s,p}(\Omega)$ as the completion of $C_0^{\infty}(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{s,p}(\Omega)}=[u]_{s,p,\Omega}. \tag{2.15}$$

Let us define weighted fractional Sobolev space on homogeneous Lie groups in the following form

$$W^{s,p,\beta}(\mathbb{G})=\{u:u\in L^p(\mathbb{G}),$$
$$[u]_{s,p,\beta}=\left(\int_{\mathbb{G}}\int_{\mathbb{G}}\frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{1}{p}}<+\infty\}, \tag{2.16}$$

where $\beta_1,\ \beta_2\in\mathbb{R}$ with $\beta=\beta_1+\beta_2$, that is, it depends on β_1 and β_2.

As above, for a Haar measurable set $\Omega\subset\mathbb{G}$, $p\geq 1$, $s\in(0,1)$ and $\beta_1,\ \beta_2\in\mathbb{R}$ with $\beta=\beta_1+\beta_2$, we define the weighted fractional Sobolev space

$$W^{s,p,\beta}(\Omega)=\{u:u\in L^p(\Omega),$$
$$[u]_{s,p,\beta,\Omega}=\left(\int_{\Omega}\int_{\Omega}\frac{|x|^{\beta_1 p}|y|^{\beta_2 p}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{1}{p}}<+\infty\}. \tag{2.17}$$

Obviously, taking $\beta = \beta_1 = \beta_2 = 0$ in (2.17), we recover (2.14).

Then, let us give main definition of the fractional p-sub-Lapalcian. For a (Haar) measurable and compactly supported function u the fractional p-sub-Laplacian $(-\Delta_p)^s$ on $\mathbb{G}$ can be defined as

$$(-\Delta_p)^s u(x) = 2 \lim_{\delta \searrow 0} \int_{\mathbb{G}\setminus B(x,\delta)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|y^{-1}x|^{Q+sp}} dy, \quad x \in \mathbb{G}, \tag{2.18}$$

where $|\cdot|$ is a quasi-norm on $\mathbb{G}$ and $B(x,\delta)$ is a quasi-ball with respect to $|\cdot|$, with radius δ centered at $x \in \mathbb{G}$. If $p = 2$, then we have $(-\Delta_2)^s = (-\Delta_s)$.

If $p > 1$, for all $\varphi \in W_0^{s,p}(\Omega)$, we have

$$\langle (-\Delta_p)^s u, \varphi \rangle := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+sp}} dxdy. \tag{2.19}$$

2.2. **Graded Lie group.** In this section, we present a brief summary of the basic definitions and properties of the graded Lie groups.

Definition 2.9 (Graded Lie group and graded Lie algebra (see e.g., [4, Definition 1.1.4] and [3, Definition 3.1.1])). A Lie algebra $\mathfrak{g}$ is called *graded* if it is endowed with a vector space decomposition (where all but finitely many of the V_j's are 0)

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j, \quad \text{s.t. } [V_i, V_j] \subset V_{i+j}. \tag{2.20}$$

Consequently, a Lie group is called *graded* if it is a connected and simply-connected Lie group whose Lie algebra is graded.

Before defining the Rockland operator, let us define Rockland condition. By π and $\widehat{\mathbb{G}}$ we define representation and unitary dual of $\mathbb{G}$, respectively and by $\mathcal{H}_\pi^\infty$ we define the smooth vectors of representation $\pi \in \widehat{\mathbb{G}}$. Let us give definition of the Rockland condition (see [3, Definition 4.1.1]):

Definition 2.10 (Rockland condition). Let A be a left-invariant differential operator on a Lie group $\mathbb{G}$. Then A satisfies the *Rockland condition* when

(Rockland condition) for each representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(A)$ is injective on $\mathcal{H}_\pi^\infty$, that is,

$$\forall v \in \mathcal{H}_\pi^\infty, \quad \pi(A)v = 0 \Rightarrow v = 0. \tag{2.21}$$

Then let us give Rockland operator on homogeneous Lie groups $\mathbb{G}$ (see e.g., [3, Definition 4.1.2]).

Definition 2.11 (Rockland operator). Let $\mathbb{G}$ be a homogeneous Lie group. A *Rockland operator* $\mathcal{R}$ on $\mathbb{G}$ is a left-invariant differential operator which is homogeneous of positive degree and satisfies the Rockland condition.

Then let us give proposition which connected homogeneous Lie groups and Rockland operators.

Proposition 2.12 ([3, Proposition 4.1.3]). *Let $\mathbb{G}$ be a homogeneous Lie group. If there exists a Rockland operator on $\mathbb{G}$ then the $\mathbb{G}$ is a graded.*

Then let us give some example for the Rockland operator on graded Lie group.

Lemma 2.13 ([3, Lemma 4.1.8])**.** *Let $\mathbb{G}$ be a graded Lie group on $\mathbb{R}^n$. We denote by $\{D_r\}_{r>0}$ the natural family of dilations on its Lie algebra $\mathfrak{g}$, and by $v_1, ..., v_n$ its weights. We fix a basis $\{X_1, ..., X_n\}$ of $\mathfrak{g}$ satisfying*

$$D_r X_j = r^{v_j} X_j, \ \ j = 1, \dots, n, \ r > 0.$$

If ν_0 is any common multiple of $v_1, \dots, v_n$, the operator

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_0}{v_j}} c_j X_j^{2\frac{\nu_0}{v_j}}, \ \ c_j = const, \tag{2.22}$$

is a Rockland operator of homogeneous degree $2\nu_0$.

By combining Proposition 2.12 and Lemma 2.13, we have that the in homogeneous Lie group $\mathbb{G}$, if there exists Rockland operator in the form (2.22) as in Lemma 2.13, then $\mathbb{G}$ is a graded. In the Chapter 5, we will widely use Rockland operator as in Lemma 2.13. Let us give definition of fractional power of the Rockland operator (see, [3, Definition 4.3.1]).

Definition 2.14. Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$. For $p \in [1, \infty)$, we denote by $\mathcal{R}_p$ the operator such that $-\mathcal{R}_p$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t > 0$, on $L^p(\mathbb{G})$.

Then let us give a definition of the Sobolev space on graded Lie groups. Assume that $\mathcal{R}$ be a positive Rockland with homogeneous degree ν and $\mathcal{R}_p$ fractional power of $\mathcal{R}$ on graded Lie group $\mathbb{G}$, which is defined in Definitions 2.11 and 2.14, respectively.

Definition 2.15 (Inhomogeneous Sobolev space ([3, Definition 4.2.2]))**.** Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$ and $s \in \mathbb{R}$. If $p \in [1, \infty)$, the Sobolev space $L^p_s(\mathbb{G})$ is the subspace of $S'(\mathbb{G})$ obtained by completion of $S(\mathbb{G})$ with respect to the Sobolev norm

$$\|f\|_{L^p_s(G)} := \|(I + \mathcal{R}_p)^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}, \quad \forall f \in S(\mathbb{G}).$$

Let us give definition of the homogeneous Sobolev space on graded Lie groups.

Definition 2.16 ([3, Definition 4.4.12])**.** Let $\mathcal{R}$ be a Rockland operator of homogeneous degree ν on a graded Lie group $\mathbb{G}$, and let $p \in (1, \infty)$. We denote by $\dot{L}^p_s(\mathbb{G})$ the space of tempered distribution obtained by the completion of $S(\mathbb{G}) \cap Dom(\mathcal{R}_p^{\frac{s}{\nu}})$ for the norm

$$\|f\|_{\dot{L}^p_s(\mathbb{G})} := \|\mathcal{R}_p^{\frac{s}{\nu}} f\|_{L^p(\mathbb{G})}, \quad \forall f \in S(\mathbb{G}) \cap Dom(\mathcal{R}_p^{\frac{s}{\nu}}).$$

Then let us give the following theorem about the independence of the spaces $L^p_s(\mathbb{G})$ and $\dot{L}^p_s(\mathbb{G})$ of a particular choice of the Rockland operator $\mathcal{R}$.

Theorem 2.17 ([3, Theorem 4.4.20])**.** *Let $\mathbb{G}$ be a graded Lie group and $p \in (1, \infty)$. The homogeneous L^p-Sobolev spaces on $\mathbb{G}$ associated with any positive Rockland operators coincide. The inhomogeneous L^p-Sobolev spaces on $\mathbb{G}$ associated with any positive Rockland operators coincide.*

Then by using last theorem, norms of the inhomogeneous and homogeneous Sobolev spaces on graded Lie groups, respectively have the following forms:

$$\|f\|_{L^p_s(\mathbb{G})} = \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{s}{\nu}} f|^p dx + \int_{\mathbb{G}} |f|^p dx\right)^{\frac{1}{p}}, \tag{2.23}$$

and

$$\|f\|_{\dot{L}^p_s(\mathbb{G})} = \left(\int_{\mathbb{G}} |\mathcal{R}^{\frac{s}{\nu}} f|^p dx\right)^{\frac{1}{p}}. \tag{2.24}$$

In this disseratation, we can use any of the notation of the Sobolev space on graded Lie groups $L^p_s(\mathbb{G}) = H^s(\mathbb{G})$.

2.3. **Stratified Lie group.** In this section, we give definitions of stratified group (homogeneous Carnot group) and basic propositions. Let us briefly recall the definition of the stratified Lie group. We refer [2], [3] and [4] for further discussions in this direction.

Definition 2.18. A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified Lie group if it satisfies the following assumptions:

(a) For some natural numbers $n_1+...+n_r = n$ the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1}\times...\times\mathbb{R}^{n_r}$ is valid, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\delta_\lambda(x) \equiv \delta_\lambda(x^{(1)}, ..., x^{(r)}) := (\lambda x^{(1)}, ..., \lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. Here $x^{(k)} \in \mathbb{R}^{n_k}$ for $k = 1, ..., r$.

(b) Let n_1 be as in (a) and let $X_1, ..., X_{n_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, ..., n_1$. Then

$$\operatorname{rank}(\operatorname{Lie}\{X_1, ..., X_{n_1}\}) = n,$$

for every $x \in \mathbb{R}^n$, i.e. the iterated commutators of $X_1, ..., X_{n_1}$ span the Lie algebra of $\mathbb{G}$.

Also, by [3] and [4] we have the following definition of the stratified Lie group:

Definition 2.19. A graded Lie algebra $\mathfrak{g}$ is called stratified if V_1 generates $\mathfrak{g}$ as an algebra. In this case, if $\mathfrak{g}$ is nilpotent of step m we have

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j, \quad \text{s.t. } [V_i, V_1] \subset V_{i+1}, \tag{2.25}$$

and the natural dilations $\mathfrak{g}$ are given by

$$D_r\left(\sum_{k=1}^{m} X_k\right) = \sum_{k=1}^{m} r^k X_k, \quad (X_k \in V_k). \tag{2.26}$$

Consequently, a Lie group is called stratified if it is connected and simply-connected Lie group whose Lie algebra is stratified.

As in homogeneous groups, by dx we understand Haar measure on stratified Lie group $\mathbb{G}$.

Then let us give as example of the stratified Lie groups which is called the Heisenberg group. Let us briefly give the definition of the Heisenberg group. By $\mathbb{H}^n := (\mathbb{R}^{2n+1}, \circ)$, we define Heisenberg group with group law:

$$\widetilde{\xi}\circ\xi' = (\widetilde{x}+x', \widetilde{y}+y', t+t'+2(x'\widetilde{y}-\widetilde{x}y')), \quad \forall\xi = (\widetilde{x}, \widetilde{y}, t) \text{ and } \forall\xi' = (x', y', t'), \tag{2.27}$$

where $\widetilde{\xi} = (\widetilde{x}, \widetilde{y}, t) \in \mathbb{R}^{2n+1}$ with $\widetilde{x} \in \mathbb{R}^n$, $\widetilde{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The family of dilations has the following form

$$\delta_\lambda(\widetilde{\xi}) := (\lambda\widetilde{x}, \lambda\widetilde{y}, \lambda^2 t), \quad \forall\lambda > 0. \tag{2.28}$$

Then, homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n+2$ and the topological dimension is $2n+1$. The Lie algebra $\mathfrak{g}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}^n$ is spanned by

$$X_i = \partial_i + 2\widetilde{y}_i\partial_t, \quad i = 1, \ldots, n,$$

$$Y_i = \partial_{n+i} - 2\widetilde{x}_i\partial_t, \quad i = 1, \ldots, n,$$

with their non-zero commutator

$$[X_i, Y_i] = -4\partial_t.$$

Let us define Sobolev space on stratified Lie groups. By the notation

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_{N_1})$$

we called (horizontal) gradient. Let Ω be an open subset $\mathbb{G}$. Let us consider Sobolev space

$$S^{1,p}(\Omega) := \{u : u \in L^p(\Omega),\ |\nabla_{\mathbb{G}} u| \in L^p(\Omega), \quad p \geq 1\}, \tag{2.29}$$

supplemented with the norm

$$\|u\|_{S^{1,p}(\Omega)} := \left(\int_\Omega |u|^p + |\nabla_{\mathbb{G}} u|^p dx \right)^{\frac{1}{p}}.$$

Then, we define the functional class $S_0^{1,p}(\Omega)$ to be the completion of $C_0^1(\Omega)$ in the norm

$$\|u\|_{S_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^p dx \right)^{\frac{1}{p}}.$$

So, the sub-Laplacian on stratified groups is given by

$$\Delta_{\mathbb{G}} := \nabla_{\mathbb{G}} \cdot \nabla_{\mathbb{G}},$$

and the p-sub-Laplacian is given by

$$\mathcal{L}_p := \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}}|^{p-2} \nabla_{\mathbb{G}}).$$

On Heisenberg group, the sub-Laplacian is given by

$$\Delta_H := \nabla_H \cdot \nabla_H,$$

where $\nabla_H = (X_1, \ldots, Y_n)$, and the p-sub-Laplacian is given by

$$\Delta_{H,p} := \nabla_H \cdot (|\nabla_H|^{p-2} \nabla_H). \tag{2.30}$$

For simplicity, throughout this dissertation we use any of the notation ∇_H and $\nabla_{\mathbb{H}^n}$ for the horizontal gradient and for the sub-Laplacian we use any of the notation Δ_H and $\Delta_{\mathbb{H}^n}$. It is well known that the class of the Heisenberg group is a subclass of the stratified Lie groups, that is, obviously, the above definition is valid for the Heisenberg group setting.

2.4. **Metric measure space, Hyperbolic space and Cartan-Hadamard manifolds.** Let us introduce, main definitions of the metric measure space, hyperbolic and Cartan-Hadamard manifolds. Definitions of this sections will widely use in the Chapter 4.

Definition 2.20 ([12])**.** Let $(\mathbb{X}, d)$ be a metric space where d is a metric and dx be a Borel measure. Then this triple $(\mathbb{X}, d, dx)$ is called metric measure space.

By [12], let us consider $(\mathbb{X}, d, dx)$ metric measure space allowing for the following polar decomposition at $a \in \mathbb{X}$: we assume that there is a locally integrable function $\lambda \in L^1_{loc}$ such that for all $f \in L^1(\mathbb{X})$ we have

$$\int_{\mathbb{X}} f(x)dx = \int_0^{\infty} \int_{\Sigma_r} f(r,\omega)\lambda(r,\omega)d\omega dr, \tag{2.31}$$

for the set $\Sigma_r = \{x \in \mathbb{X} : d(x,a) = r\} \subset \mathbb{X}$ with a measure on it denoted by $d\omega$, and $(r,\omega) \to a$ as $r \to 0$. This polar decomposition will play a key role in the proof of our results in Chapter 4.

Let us give definition of the hyperbolic space.

Definition 2.21. The hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ is a complete and simply connected Riemannian manifold having constant sectional curvature equal to -1.

Let us denote that by $d(0,x)$ the hyperbolic distance in the ball model between the origin and x in the following form: $d(0,x) = \ln \frac{1+|x|}{1-|x|}$. So then let us give definition of the Cartan-Hadamard manifolds:

Definition 2.22 ([12])**.** Let K_M be the sectional curvature on (M,g). A Riemannian manifold (M,g) is called a Cartan-Hadamard manifold if it is complete, simply connected and has non-positive sectional curvature, i.e., the sectional curvature $K_M \leq 0$ along each plane section at each point of M.

By [12], the condition (2.31) is rather general since we allow the function λ to depend on the whole variable $x = (r,\omega)$. The reason to assume (2.31) is that since $\mathbb{X}$ does not have to have a differentiable structure, the function $\lambda(r,\omega)$ can not be in general obtained as the Jacobian of the polar change of coordinates. However, if such a differentiable structure exists on $\mathbb{X}$, the condition (2.31) can be obtained as the standard polar decomposition formula. In particular, let us give several examples of $\mathbb{X}$ for which the condition (2.31) is satisfied with different expressions for $\lambda(r,\omega)$:

(I) Euclidean space $\mathbb{R}^n$: $\lambda(r,\omega) = r^{n-1}$.

(II) Homogeneous groups: $\lambda(r,\omega) = r^{Q-1}$, where Q is the homogeneous dimension of the group. Such groups have been consistently developed by Folland and Stein [1], see also an up-to-date exposition in [3].

(III) Hyperbolic spaces $\mathbb{H}^n$: $\lambda(r,\omega) = (\sinh r)^{n-1}$.

(IV) Cartan-Hadamard manifolds: Let us fix a point $a \in M$ and denote by $\rho(x) = d(x,a)$ the geodesic distance from x to a on M. The exponential map $\exp_a : T_aM \to M$ is a diffeomorphism, see e.g. Helgason [60]. Let $J(\rho,\omega)$ be the density function on M. Then we have the following polar decomposition:

$$\int_M f(x)dx = \int_0^{\infty} \int_{\mathbb{S}^{n-1}} f(\exp_a(\rho\omega))J(\rho,\omega)\rho^{n-1}d\rho d\omega,$$

so that we have (2.31) with $\lambda(\rho, \omega) = J(\rho, \omega)\rho^{n-1}$.

3. Direct Inequalities

In this chapter, we show basic direct fractional functional and geometric inequalities on homogeneous Lie group.

3.1. **Fractional Hardy inequality.**

In this section, we show fractional Hardy inequality. For showing fractional Hardy inequality, we need some preliminary results.

Lemma 3.1 ([33], Lemma 2.6)**.** *Assume that $p>1$, then for all $t\in[0,1]$ and $a\in\mathbb{C}$, we have*

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t). \tag{3.1}$$

In the all following lemma, we assume that $Q>2$, $p>1$ and $s\in(0,1)$ be such that $Q>sp$.

Lemma 3.2 (Picone-type inequality)**.** *Let $\omega\in W_0^{s,p}(\Omega)$ be $\omega>0$ in $\Omega\subset\mathbb{G}$ and suppose that $(-\Delta_p)^s\omega=\nu>0$ with $\nu\in L^1_{loc}(\Omega)$, then for all $u\in C_0^\infty(\Omega)$, we have*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}}dxdy \geq \left\langle (-\Delta_p)^s\omega, \frac{|u|^p}{\omega^{p-1}}\right\rangle. \tag{3.2}$$

Proof. Proof of this lemma is based [34] and [61]. By setting $v=\frac{|u|^p}{|\omega|^{p-1}}$ and $k(x,y)=\frac{1}{|y^{-1}x|^{Q+ps}}$, then we obtain

$$\begin{aligned}\langle(-\Delta_p)^s\omega,v\rangle_{L^2(\Omega)} &= \int_\Omega v(x)dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy\\ &= \int_\Omega \frac{|u|^p}{|\omega|^{p-1}}dx\int_\Omega |\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dy,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is inner product in $L^2(\Omega)$, By using the definition of quasi-norm we have $|x^{-1}|=|x|$ for all $x\in\mathbb{G}$. Then we get

$$\begin{aligned}k(x,y) &= \frac{1}{|y^{-1}x|^{Q+ps}} = \frac{1}{|z|^{Q+ps}} = \frac{1}{|z^{-1}|^{Q+ps}}\\ &= \frac{1}{|(y^{-1}x)^{-1}|^{Q+ps}} = \frac{1}{|x^{-1}y|^{Q+ps}} = k(y,x),\end{aligned}$$

for all $x,y\in\mathbb{G}$. By using $k(x,y)$ is symmetric, we obtain that

$$\langle(-\Delta_p)^s\omega,v\rangle_{L^2(\Omega)} =$$

$$\frac{1}{2}\int_\Omega\int_\Omega\left(\frac{|u(x)|^p}{|\omega(x)|^{p-1}}-\frac{|u(y)|^p}{|\omega(y)|^{p-1}}\right)|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y))k(x,y)dydx.$$

Let $g=\frac{u}{\omega}$ and

$$R(x,y)=|u(x)-u(y)|^p-(|g(x)|^p\omega(x)-|g(y)|^p\omega(y))|\omega(x)-\omega(y)|^{p-2}(\omega(x)-\omega(y)),$$

then we have

$$\langle(-\Delta_p)^s\omega,v\rangle_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\int_\Omega R(x,y)k(x,y)dydx=\frac{1}{2}\int_\Omega\int_\Omega|u(x)-u(y)|^pk(x,y)dydx.$$

By the symmetry argument, we can assume that $\omega(x) \geq \omega(y)$. By using Lemma 3.1 with $t = \frac{\omega(y)}{\omega(x)}$ and $a = \frac{g(x)}{g(y)}$ and we establish that $R(x, y) \geq 0$. Thus, we have proved the inequality

$$\langle (-\Delta_p)^s \omega, v \rangle_{L^2(\Omega)} \leq \frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|y^{-1}x|^{Q+ps}} dy dx.$$

Lemma 3.2 is proved. ∎

Lemma 3.3. *Let $\omega = |x|^{-\gamma}$ with $\gamma \in \left(0, \frac{Q-ps}{p-1}\right)$ where $Q > sp$, then there exists a positive constant $\mu(\gamma) > 0$ such that*

$$(-\Delta_p)^s (|x|^{-\gamma}) = \mu(\gamma) \frac{1}{|x|^{ps+\gamma(p-1)}} \text{ a.e. in } \mathbb{G} \setminus \{0\}. \tag{3.3}$$

Proof. Let us set $r = |x|$ and $\rho = |y|$ with $x = rx'$ and $y = \rho y'$ where $|x'| = |y'| = 1$. Then by using polar decomposition (see (2.11)), we have

$$\begin{aligned}
&(-\Delta_p)^s \omega \\
&= \int_0^{+\infty} |r^{-\gamma} - \rho^{-\gamma}|^{p-2} (r^{-\gamma} - \rho^{-\gamma}) \rho^{Q-1} \left(\int_{\mathfrak{S}} \frac{d\sigma(y')}{|(\rho y')^{-1}(rx')|^{Q+ps}} \right) d\rho \\
&= \frac{1}{|x|^{ps+\gamma(p-1)}} \int_0^{+\infty} \left| 1 - \frac{\rho^{-\gamma}}{r^{-\gamma}} \right|^{p-2} \times \\
&\times \left(1 - \frac{\rho^{-\gamma}}{r^{-\gamma}} \right) \frac{\rho^{Q-1}}{r^Q} \left(\int_{\mathfrak{S}} \frac{d\sigma(y')}{|\left(\frac{\rho}{r} y'\right)^{-1} x'|^{Q+ps}} \right) d\rho.
\end{aligned}$$

Let $\widetilde{\rho} = \frac{\rho}{r}$ and $L(\widetilde{\rho}) = \int_{\mathfrak{S}} \frac{d\sigma(y')}{|(\widetilde{\rho} y')^{-1} x')|^{Q+ps}}$, we get

$$(-\Delta_p)^s \omega = \frac{1}{|x|^{ps+\gamma(p-1)}} \int_0^{+\infty} |1 - \widetilde{\rho}^{-\gamma}|^{p-2} (1 - \widetilde{\rho}^{-\gamma}) L(\widetilde{\rho}) \widetilde{\rho}^{Q-1} d\widetilde{\rho}.$$

Then it easy to see

$$\mu(\gamma) = \int_0^{+\infty} \phi(\widetilde{\rho}) d\widetilde{\rho} \tag{3.4}$$

with $\phi(\widetilde{\rho}) = |1 - \widetilde{\rho}^{-\gamma}|^{p-2} (1 - \widetilde{\rho}^{-\gamma}) L(\widetilde{\rho}) \widetilde{\rho}^{Q-1}$.

We need to show that $\mu(\gamma)$ is a positive and bounded. Firstly, let us show boundedness of $\mu(\gamma)$. We get

$$\mu(\gamma) = \int_0^1 \phi(\widetilde{\rho}) d\widetilde{\rho} + \int_1^{+\infty} \phi(\widetilde{\rho}) d\widetilde{\rho} = I_1 + I_2. \tag{3.5}$$

By changing to the new variable $\zeta = \frac{1}{\widetilde{\rho}}$ we obtain $L(\widetilde{\rho}) = L\left(\frac{1}{\zeta}\right) = \zeta^{Q+ps} L(\zeta)$ for any $\zeta > 0$. Thus, we establish

$$\mu(\gamma) = \int_1^{+\infty} (\rho^{-\gamma} - 1)^{p-1} (\rho^{Q-1-\gamma(p-1)} - \rho^{ps-1}) L(\rho) d\rho. \tag{3.6}$$

For $\rho \to 1$ we get

$$(\rho^{-\gamma} - 1)^{p-1} (\rho^{Q-1-\gamma(p-1)} - \rho^{ps-1}) L(\rho) \simeq (\rho - 1)^{-1-ps+p} \in L^1(1, 2). \tag{3.7}$$

Similarly, for $\rho \to \infty$ we have

$$(\rho^{-\gamma}-1)^{p-1}(\rho^{Q-1-\gamma(p-1)}-\rho^{ps-1})L(\rho) \simeq \rho^{-1-ps} \in L^1(2,\infty). \tag{3.8}$$

It means we show that $\mu(\gamma)$ is bounded. By (3.6) with $\gamma \in \left(0, \frac{Q-ps}{p-1}\right)$ we see that $\mu(\gamma)$ is positive.

Lemma 3.3 is proved. □

Finally, as a result we obtain the following analogue of the fractional Hardy inequality on $\mathbb{G}$.

Theorem 3.4 (Fractional Hardy inequality)**.** *Assume $Q > 2$, $p > 1$ and $s \in (0,1)$ be such that $Q > sp$. Then for all $u \in C_0^\infty(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{sp}} dx \le C[u]^p_{s,p}, \tag{3.9}$$

where C is positive constant.

Proof. Let $u \in C_0^\infty(\mathbb{G})$ and $\gamma < \frac{Q-ps}{p-1}$. By using Lemma 3.3 and Lemma 3.2 we have

$$\begin{aligned}\frac{1}{2}[u]^p_{s,p} = \frac{1}{2}\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+ps}} dxdy &\ge \left\langle (-\Delta_p)^s(|x|^{-\gamma}), \frac{|u(x)|^p}{|x|^{-\gamma(p-1)}} \right\rangle \\ &= \mu(\gamma)\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{ps}} dx,\end{aligned} \tag{3.10}$$

completing proof. □

3.2. **Fractional Sobolev inequality.** In this section we prove fractional Sobolev inequality on the homogeneous Lie groups.

For showing an analogue of the fractional Sobolev inequality, firstly we need show some preliminary results.

Lemma 3.5. *Let $p > 1$, $s \in (0,1)$ and $K \subset \mathbb{G}$ be Haar measurable set. Fix $x \in \mathbb{G}$ and a quasi-norm $|\cdot|$ on $\mathbb{G}$, then we have*

$$\int_{K^c} \frac{dy}{|y^{-1}x|^{Q+sp}} \ge C|K|^{-sp/Q}, \tag{3.11}$$

where $C = C(Q,s,p)$ is a positive constant, $K^c = \mathbb{G}\setminus K$ and $|K|$ is the Haar measure of K.

Proof. By setting $\delta := \left(\frac{Q|K|}{\omega_Q}\right)^{1/Q}$, where ω_Q is a surface measure of the unit quasi-ball on $\mathbb{G}$ and let us fix $x \in \mathbb{G}$ such that $K \cap B(x,\delta) \neq \emptyset$ where $B(x,\delta)$ is a quasi-ball centered at x with radius δ. Then, we get

$$\begin{aligned}|K^c \cap B(x,\delta)| &= |B(x,\delta)| - |K \cap B(x,\delta)| \\ &= |K| - |K \cap B(x,\delta)| = |K \cap B^c(x,\delta)|,\end{aligned} \tag{3.12}$$

where $|\cdot|$ is the Haar measure on $\mathbb{G}$. Then,

$$\begin{aligned}\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}} &= \int_{K^c\cap B(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &\geq \int_{K^c\cap B(x,\delta)}\frac{dy}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

By using (3.12) we get

$$\begin{aligned}\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}} &\geq \frac{|K^c\cap B(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \frac{|K\cap B^c(x,\delta)|}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \int_{K\cap B^c(x,\delta)}\frac{dy}{\delta^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &\geq \int_{K\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}+\int_{K^c\cap B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}\\ &= \int_{B^c(x,\delta)}\frac{dy}{|y^{-1}x|^{Q+sp}}.\end{aligned}$$

By using the polarisation formula (2.11) with centered at x, we have

$$\int_{K^c}\frac{dy}{|y^{-1}x|^{Q+sp}}\geq C|K|^{-sp/Q}. \tag{3.13}$$

$\square$

Lemma 3.6 ([39], Lemma 6.2). *Fix $T>1$. Let $p>1$ and $s\in(0,1)$ be such that $Q>sp$, $m\in\mathbb{Z}$ and a_k be a bounded, decreasing, nonnegative sequence with $a_k=0$ for any $k\geq m$. Then*

$$\sum_{k\in\mathbb{Z}}a_k^{(Q-sp)/Q}T^k\leq C\sum_{k\in\mathbb{Z},\,a_k\neq 0}a_{k+1}a_k^{-sp/Q}T^k,$$

for a positive constant $C=C(Q,s,p,T)>0$.

Lemma 3.7. *Suppose that $p>1$, $s\in(0,1)$, $Q>sp$ and $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Assume that $u\in L^\infty(\mathbb{G})$ be compactly supported and $a_k:=|\{|u|>2^k\}|$ for any $k\in\mathbb{Z}$. Then,*

$$C\sum_{k\in\mathbb{Z},\,a_k\neq 0}a_{k+1}a_k^{-sp/Q}2^{kp}\leq [u]^p_{s,p}, \tag{3.14}$$

where $C=C(Q,p,s)$ is a positive constant and $[u]_{s,p}$ is defined by (2.12).

Proof. Let us define

$$A_k:=\{|u|>2^k\},\ k\in\mathbb{Z}, \tag{3.15}$$

and

$$D_k:=A_k\setminus A_{k+1}=\{2^k<|u|\leq 2^{k+1}\}\ \text{ and }\ d_k=|D_k|. \tag{3.16}$$

Since $A_{k+1} \subseteq A_k$, it is easy to see

$$a_{k+1} \leq a_k. \tag{3.17}$$

By the assumption $u \in L^\infty(\mathbb{G})$ is compactly supported, a_k and d_k are bounded and vanish when k is large enough. Also, we notice that the D_k's are disjoint, therefore,

$$\bigcup_{l\in\mathbb{Z},\ l\leq k} D_l = A^c_{k+1} \tag{3.18}$$

and

$$\bigcup_{l\in\mathbb{Z},\ l\geq k} D_l = A_k. \tag{3.19}$$

By using (3.19) we establish that

$$\sum_{l\in\mathbb{Z},\ l\geq k} d_l = a_k \tag{3.20}$$

and

$$d_k = a_k - \sum_{l\in\mathbb{Z},\ l\geq k+1} d_l. \tag{3.21}$$

From a_k and d_k are bounded and vanish when k is large enough, (3.20) and (3.21) are convergent. Let us define the convergent series

$$S := \sum_{l\in\mathbb{Z},\ a_{l-1}\neq 0} 2^{lp} a_{l-1}^{-sp/Q} d_l. \tag{3.22}$$

We have that $D_k \subseteq A_k \subseteq A_{k-1}$, then, $a_{i-1}^{-sp/Q} d_l \leq a_{i-1}^{-sp/Q} a_{l-1}$. Thus,

$$\{(i,l) \in \mathbb{Z} \text{ s.t. } a_{i-1} \neq 0 \text{ and } a_{i-1}^{-sp/Q} d_l \neq 0\} \subseteq \{(i,l) \in \mathbb{Z} \text{ s.t. } a_{l-1} \neq 0\}. \tag{3.23}$$

By combining (3.23) and (3.17), we compute that

$$\begin{aligned}
&\sum_{i\in\mathbb{Z},\ a_{i-1}\neq 0}\ \sum_{l\in\mathbb{Z},\ l\geq i+1} 2^{ip} a_{i-1}^{-sp/Q} d_l = \sum_{i\in\mathbb{Z},\ a_{i-1}\neq 0}\ \sum_{l\in\mathbb{Z},\ l\geq i+1,\ a^{sp/Q} d_l\neq 0} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
&\leq \sum_{i\in\mathbb{Z}}\ \sum_{l\in\mathbb{Z},\ l\geq i+1,\ a_{l-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} d_l = \sum_{l\in\mathbb{Z},\ a_{l-1}\neq 0}\ \sum_{i\in\mathbb{Z},\ i\leq l-1} 2^{ip} a_{i-1}^{-sp/Q} d_l \\
&\leq \sum_{l\in\mathbb{Z},\ a_{l-1}\neq 0}\ \sum_{i\in\mathbb{Z},\ i\leq l-1} 2^{ip} a_{l-1}^{-sp/Q} d_l = \sum_{l\in\mathbb{Z},\, a_{l-1}\neq 0}\ \sum_{k=0}^{+\infty} 2^{p(l-1-k)} a_{l-1}^{-sp/Q} d_l \leq S.
\end{aligned} \tag{3.24}$$

Notice that

$$||u(x)| - |u(y)|| \leq |u(x) - u(y)|, \quad \forall x, y \in \mathbb{G}.$$

By setting $i \in \mathbb{Z}$ and $x \in D_i$, then for all $j \in \mathbb{Z}$ with $j \leq i-2$, for any $y \in D_j$ using the last inequality, we have that

$$|u(x) - u(y)| \geq 2^i - 2^{j+1} \geq 2^i - 2^{i-1} \geq 2^{i-1}$$

and using (3.18), we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dy \ge 2^{(i-1)p}\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{dy}{|y^{-1}x|^{Q+sp}}$$
$$= 2^{(i-1)p}\int_{A^c_{i-1}}\frac{dy}{|y^{-1}x|^{Q+sp}}. \quad (3.25)$$

By combining (3.25) and Lemma 3.5, we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dy \ge C2^{ip}a_{i-1}^{-sp/Q},$$

where C is a positive constant. It means, for any $i\in\mathbb{Z}$, we get

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge C2^{ip}a_{i-1}^{-sp/Q}d_i. \quad (3.26)$$

By combinig (3.26) and (3.21) we obtain that

$$\sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy$$
$$\ge C\left(2^{ip}a_{i-1}^{-sp/Q}a_i-\sum_{l\in\mathbb{Z},\, l\ge i+1}2^{ip}a_{i-1}^{-sp/Q}d_l\right). \quad (3.27)$$

From (3.26) and (3.22) we obtain that

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy$$
$$\ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}2^{ip}a_{i-1}^{-sp/Q}d_i \ge C\,S. \quad (3.28)$$

Then, by using (3.24), (3.27) and (3.28), we have

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}2^{ip}a_{i-1}^{-sp/Q}a_i$$
$$-C\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{l\in\mathbb{Z},\, l\ge i+1}2^{ip}a_{i-1}^{-sp/Q}d_l \ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}2^{ip}a_{i-1}^{-sp/Q}a_i - C\,S$$
$$\ge C\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}2^{ip}a_{i-1}^{-sp/Q}a_i-\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy.$$

Thus,

$$\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}\ \sum_{j\in\mathbb{Z},\, j\le i-2}\int_{D_i}\int_{D_j}\frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \ge \frac{C}{2}\sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0}2^{ip}a_{i-1}^{-sp/Q}a_i, \quad (3.29)$$

for a constant $C > 0$. By using symmetry property and (3.29), we obtain that

$$
\begin{aligned}
[u]^p_{s,p} &= \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = \sum_{i,j\in\mathbb{Z}} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq 2 \sum_{i,j\in\mathbb{Z},\, j<i} \int_{D_i}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq 2 \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} \; \sum_{j\in\mathbb{Z},\; j\leq i-2} \int_{Di}\int_{D_j} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&\geq C \sum_{i\in\mathbb{Z},\, a_{i-1}\neq 0} 2^{ip} a_{i-1}^{-sp/Q} a_i.
\end{aligned}
$$

Lemma 3.7 is proved. ☐

Lemma 3.8. *Assume that $1 < p < \infty$ and $u : \mathbb{G} \to \mathbb{R}$ be a measurable function. For any $n \in \mathbb{R}$*

$$u_n := \max\{\min\{u(x), n\}, -n\}, \quad \textit{for any } x \in \mathbb{G}. \tag{3.30}$$

Then,

$$\lim_{n\to+\infty} \|u_n\|_{L^p(\mathbb{G})} = \|u\|_{L^p(\mathbb{G})}.$$

Proof. The proof is the same as in [39, Lemma 6.4]. ☐

Then, by using the above lemmas we show the following analogue of the fractional Sobolev inequality on $\mathbb{G}$:

Theorem 3.9 (Fractional Sobolev inequality)**.** *Let $p > 1$, $s \in (0, 1)$, $Q > sp$ be such that $p^* = p^*(Q, s, p) = \frac{Qp}{Q-sp}$. Let $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$, we have*

$$\|u\|_{L^{p^*}(\mathbb{G})} \leq C[u]_{s,p}, \tag{3.31}$$

where $C = C(Q, p, s) > 0$.

Proof. Firstly, assume that $[u]_{s,p}$ (Gagliardo seminorm) is bounded, i.e.,

$$[u]^p_{s,p} = \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy < +\infty. \tag{3.32}$$

and we assume that $u \in L^\infty(\mathbb{G})$.

If (3.32) is executed for bounded functions, this is also true for the function u_n obtained by cutting the function u at levels $-n$ and n levels. Then, by combining Lemma 3.8 and (3.32) with the dominated convergence theorem, we have that

$$
\begin{aligned}
\lim_{n\to+\infty} [u_n]^p_{s,p} &= \lim_{n\to+\infty} \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u_n(x)-u_n(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy \\
&= \int_{\mathbb{G}}\int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy = [u]^p_{s,p}.
\end{aligned} \tag{3.33}
$$

As in Lemma 3.7 we define a_k and A_k, so we have

$$||u||_{L^{p^*}(\mathbb{G})} = \left(\sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} |u(x)|^{p^*} dx\right)^{1/p^*} \leq \left(\sum_{k\in\mathbb{Z}} \int_{A_k\setminus A_{k+1}} 2^{(k+1)p^*} dx\right)^{1/p^*}$$

$$\leq \left(\sum_{k\in\mathbb{Z}} 2^{(k+1)p^*} a_k \right)^{1/p^*}. \tag{3.34}$$

Therefore, by combining Lemma 3.6 with $p/p^* = 1 - sp/Q < 1$ and $T = 2^p$, obtain

$$\|u\|^p_{L^{p^*}(\mathbb{G})} \leq 2^p \left(\sum_{k\in\mathbb{Z}} 2^{kp^*} a_k \right)^{p/p^*} \leq 2^p \sum_{k\in\mathbb{Z}} 2^{kp} a_k^{(Q-sp)/Q}$$

$$\leq C \sum_{k\in\mathbb{Z},\, a_k\neq 0} 2^{kp} a_k^{-sp/Q} a_{k+1} \tag{3.35}$$

for a positive constant $C = C(Q,p,s,q) > 0$. By using Lemma 3.7 get

$$\|u\|^p_{L^{p^*}(\mathbb{G})} \leq C \sum_{k\in\mathbb{Z},\, a_k\neq 0} 2^{kp} a_k^{-sp/Q} a_{k+1} \leq C \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}} dxdy = C[u]^p_{s,p}, \tag{3.36}$$

completing the proof. □

3.3. **Fractional Hardy-Sobolev inequality.** In this section, we prove fractional Hardy-Sobolev inequality. We generalise both above inequalities, so the unified extension with arbitrary quasi-norm gives new inequalities even in the Euclidean (Abelian) case.

Theorem 3.10 (Fractional Hardy-Sobolev inequality)**.** *Suppose that $p > 1$, $s \in (0,1)$, $Q > 2$, $0 < \beta < sp$ and $Q > sp$ be such that $p^*_{s,\beta} = \frac{p(Q-\beta)}{Q-sp}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$ of $\mathbb{G}$, we have*

$$\left(\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\beta}} dx \right)^{\frac{1}{p^*_{s,\beta}}} \leq C[u]_{s,p}, \tag{3.37}$$

where C is a positive constant.

Proof. By using Hölder's inequality with $\frac{\beta}{sp} + \frac{sp-\beta}{sp} = 1$, we get

$$\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\alpha}} dx = \int_{\mathbb{G}} \frac{|u(x)|^{\frac{\beta}{s}} |u(x)|^{p^*_{s,\beta}-\frac{\beta}{s}}}{|x|^{\beta}} dx$$

$$\leq \left(\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{\frac{\beta}{sp}} \left(\int_{\mathbb{G}} |u(x)|^{\left(p^*_{s,\beta}-\frac{\beta}{s}\right)\frac{sp}{sp-\beta}} dx \right)^{\frac{sp-\beta}{sp}}. \tag{3.38}$$

By some calculation, we have

$$\left(p^*_{s,\beta} - \frac{\beta}{s} \right) \frac{sp}{sp-\beta} = \left(\frac{p(Q-\beta)}{Q-sp} - \frac{\beta}{s} \right) \frac{sp}{sp-\beta}$$

$$= \frac{Qsp - \beta sp - Q\beta + \beta sp}{s(Q-sp)} \frac{sp}{sp-\beta} = \frac{Qp}{Q-sp} = p^*,$$

where p^* is the Sobolev exponent. By combining the fractional Hardy and Sobolev inequalities, that is, Theorems 3.4 and 3.9, we establish

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\beta}} dx &\leq \left(\int_{\mathbb{G}} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{\frac{\beta}{sp}} \left(\int_{\mathbb{G}} |u(x)|^{p^*} dx \right)^{\frac{sp-\beta}{sp}} \\
&\overset{(3.9)}{\leq} C[u]_{s,p}^{\frac{\beta}{s}} \left(\int_{\mathbb{G}} |u(x)|^{p^*} dx \right)^{\frac{sp-\beta}{sp}} \\
&\overset{(3.31)}{\leq} C[u]_{s,p}^{\frac{\beta}{s}} [u]_{s,p}^{\frac{sp-\beta}{sp} p^*} \\
&= C[u]_{s,p}^{\frac{\beta}{s} + \frac{sp-\beta}{sp} p^*}.
\end{aligned} \tag{3.39}$$

Let us compute the exponent of the last term

$$\begin{aligned}
\frac{\beta}{s} + \frac{sp-\beta}{sp} p^* &= \frac{\beta}{s} + \frac{sp-\beta}{sp} \frac{Qp}{Q-sp} \\
= \frac{1}{s} \left(\frac{\beta Q - \beta sp + Qsp - \beta Q}{Q-sp} \right) &= \frac{1}{s} \frac{sp(Q-\beta)}{Q-sp} = p^*_{s,\beta}.
\end{aligned}$$

Finally, we have

$$\int_{\mathbb{G}} \frac{|u(x)|^{p^*_{s,\beta}}}{|x|^{\beta}} dx \leq C[u]_{s,p}^{\frac{\beta}{s} + \frac{sp-\beta}{sp} p^*} = C[u]_{s,p}^{p^*_{s,\beta}},$$

completing the proof. □

Corollary 3.11. *In Theorem 3.10, by setting $\beta = 0$, we obtain the fractional Sobolev inequality (3.31).*

Corollary 3.12. *When $\beta = sp$ in Theorem 3.10, we have the fractional Hardy inequality (3.9).*

Remark 3.13. *In the Abelian case $(\mathbb{R}^N, +)$, $Q = N$ with $|\cdot| = |\cdot|_E$ where $|\cdot|_E$ is the standard Euclidean distance, (3.37) implies the fractional Hardy-Sobolev inequality on $\mathbb{R}^N$ (see [62]). Moreover, the inequality is valid for any quasi-norm, not necessarily the Euclidean one. Therefore, even in the Abelian (Euclidean) case it extends the results of [62].*

3.4. **Fractional Gagliardo-Nirenberg inequality.** In this section we show fractional Gagliardo-Nirenberg inequality on homogeneous Lie groups. One of the generalisation of the fractional Sobolev inequality is the fractional Gagliardo-Nirenberg inequality.

Theorem 3.14. *Suppose that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, $Q > sp$ and*

$$\frac{1}{\tau} = a \left(\frac{1}{p} - \frac{s}{Q} \right) + \frac{1-a}{\alpha}.$$

Then,

$$\|u\|_{L^{\tau}(\mathbb{G})} \leq C[u]_{s,p}^{a} \|u\|_{L^{\alpha}(\mathbb{G})}^{1-a}, \ \forall\, u \in C_c^1(\mathbb{G}), \tag{3.40}$$

where $C = C(s,p,Q,a,\alpha) > 0$.

Proof of Theorem 3.14. By using the Hölder inequality with $\frac{1}{\tau}=a\left(\frac{1}{p}-\frac{s}{Q}\right)+\frac{1-a}{\alpha}$ we establish

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})}=\int_{\mathbb{G}}|u|^{\tau}dx=\int_{\mathbb{G}}|u|^{a\tau}|u|^{(1-a)\tau}dx\leq\|u\|^{a\tau}_{L^{p^*}(\mathbb{G})}\|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})},\tag{3.41}$$

where $p^*=\frac{Qp}{Q-sp}$. By combining (3.41) and the fractional Sobolev inequality (Theorem 3.9), we have

$$\|u\|^{\tau}_{L^{\tau}(\mathbb{G})}\leq\|u\|^{a\tau}_{L^{p^*}(\mathbb{G})}\|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})}\leq C[u]^{a\tau}_{s,p}\|u\|^{(1-a)\tau}_{L^{\alpha}(\mathbb{G})},$$

that is,

$$\|u\|_{L^{\tau}(\mathbb{G})}\leq C[u]^{a}_{s,p}\|u\|^{1-a}_{L^{\alpha}(\mathbb{G})},\tag{3.42}$$

where C is a positive constant independent of u. Theorem 3.14 is proved. ☐

Remark 3.15. *In the Abelian case $(\mathbb{R}^N,+)$ with the standard Euclidean distance instead of the quasi-norm and $s\to 1^-$, from Theorem 3.14 we get the Gagliardo-Nirenberg inequality which was proved in* [63] *and* [64].

Remark 3.16. *In the Abelian case $(\mathbb{R}^N,+)$ with the standard Euclidean distance instead of the quasi-norm, from Theorem 3.14 we get the fractional Gagliardo-Nirenberg inequality which was showed in* [41].

3.5. **Fractional Caffarelli-Kohn-Nirenberg inequality.** In this section we prove the weighted fractional Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.

Let us give some notations. The mean of a function u is defined by

$$u_{\Omega}=\fint_{\Omega}u(x)dx=\frac{1}{|\Omega|}\int_{\Omega}udx,\quad u\in L^1(\Omega),\tag{3.43}$$

where $|\Omega|$ is the Haar measure of $\Omega\subset\mathbb{G}$.

We will also use the decomposition of $\mathbb{G}$ into quasi-annuli A_k defined by

$$A_k:=\{x\in\mathbb{G}:\ 2^k\leq|x|<2^{k+1}\},\tag{3.44}$$

where $|x|$ is a quasi-norm on $\mathbb{G}$.

To show the fractional Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$ we will use the fractional Gagliardo-Nirenberg inequality (Theorem 3.14) in the proof of the following lemma.

Lemma 3.17. *Suppose that $Q\geq 2$, $s\in(0,1)$, $p>1$, $\alpha\geq 1$, $\tau>0$, $a\in(0,1]$ and*

$$\frac{1}{\tau}\geq a\left(\frac{1}{p}-\frac{s}{Q}\right)+\frac{1-a}{\alpha}.$$

Assume that $\lambda>0$ and $0<r<R$ and set

$$\Omega=\{x\in\mathbb{G}:\ \lambda r<|x|<\lambda R\}.$$

Then, for every $u\in C^1(\overline{\Omega})$, we have

$$\left(\fint_{\Omega}|u-u_{\Omega}|^{\tau}dx\right)^{\frac{1}{\tau}}\leq C_{r,R}\lambda^{\frac{a(sp-Q)}{p}}[u]^{a}_{s,p,\Omega}\left(\fint_{\Omega}|u|^{\alpha}dx\right)^{\frac{1-a}{\alpha}},\tag{3.45}$$

where $C_{r,R}$ is a positive constant independent of u and λ.

Proof of Lemma 3.17. Without loss of generality, we suppose that $0 < s' \leq s$ and $\tau' \geq \tau$ are such that

$$\frac{1}{\tau'} = a\left(\frac{1}{p} - \frac{s'}{Q}\right) + \frac{1-a}{\alpha},$$

and $\lambda = 1$, then let Ω_1 be

$$\Omega_1 = \{x \in \mathbb{G} : \ r < |x| < R\}.$$

By combining the fractional Gagliardo-Nirenberg inequality (see, Theorem 3.14), Jensen's inequality and $[u]_{s',p,\Omega} \leq C[u]_{s,p,\Omega}$, we establish

$$\begin{aligned}
\left(\fint_{\Omega_1} |u - u_{\Omega_1}|^\tau dx\right)^{\frac{1}{\tau}} &= \frac{1}{|\Omega_1|^{\frac{1}{\tau}}}\|u - u_{\Omega_1}\|_\tau \\
&\leq C_{r,R}\|u - u_{\Omega_1}\|_{L^{\tau'}(\Omega_1)} \leq C_{r,R}[u - u_{\Omega_1}]^a_{s',p,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(x) - u_{\Omega_1} - u(y) + u_{\Omega_1}|^p}{|y^{-1}x|^{Q+s'p}} dxdy\right)^{\frac{a}{p}} \|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}[u]^a_{s,p,\Omega_1}\|u\|^{1-a}_{L^\alpha(\Omega_1)} \\
&\leq C_{r,R}[u]^a_{s,p,\Omega_1}\left(\fint_{\Omega_1} |u|^\alpha dx\right)^{\frac{1-a}{\alpha}},
\end{aligned} \tag{3.46}$$

where $C_{r,R} > 0$. By setting $u(\lambda x)$ instead of $u(x)$, we have

$$\begin{aligned}
\left(\fint_{\Omega_1} \left|u(\lambda x) - \fint_{\Omega_1} u(\lambda x)dx\right|^\tau dx\right)^{\frac{1}{\tau}} &\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1} \frac{|u(\lambda x) - u(\lambda y)|^p}{|y^{-1}x|^{Q+sp}} dxdy\right)^{\frac{a}{p}} \\
&\times \left(\frac{1}{|\Omega_1|}\int_{\Omega_1} |u(\lambda x)|^\alpha dx\right)^{\frac{1-a}{\alpha}}.
\end{aligned} \tag{3.47}$$

Then by using (3.46) and Proposition 2.4, we calculate

$$
\begin{aligned}
&\left(\fint_{\Omega}\left|u(x)-\fint_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(x)-\frac{1}{|\Omega|}\int_{\Omega}u(x)dx\right|^{\tau}dx\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega|}\int_{\Omega}\left|u(\lambda y)-\frac{1}{|\Omega|}\int_{\Omega}u(\lambda y)d(\lambda y)\right|^{\tau}d(\lambda y)\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}\left|u(\lambda y)-\frac{\lambda^Q}{\lambda^Q|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}\\
&=\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\left|u(\lambda y)-\frac{1}{|\Omega_1|}\int_{\Omega_1}u(\lambda y)dy\right|^{\tau}dy\right)^{\frac{1}{\tau}}\\
&\leq C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{|u(\lambda x)-u(\lambda y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega_1}\int_{\Omega_1}\frac{\lambda^{2Q}\lambda^{Q+sp}|u(\lambda x)-u(\lambda y)|^p}{\lambda^{2Q}\lambda^{Q+sp}|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega_1|}\int_{\Omega_1}\frac{\lambda^Q}{\lambda^Q}|u(\lambda x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(\lambda x)-u(\lambda y)|^p}{|(\lambda y)^{-1}(\lambda x)|^{Q+sp}}d(\lambda x)d(\lambda y)\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(\lambda x)|^{\alpha}d(\lambda x)\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\left(\int_{\Omega}\int_{\Omega}\frac{\lambda^{sp-Q}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a}{p}}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}\\
&=C_{r,R}\lambda^{\frac{a(sp-Q)}{p}}[u]^a_{s,p,\Omega}\left(\frac{1}{|\Omega|}\int_{\Omega}|u(x)|^{\alpha}dx\right)^{\frac{1-a}{\alpha}},
\end{aligned}
\tag{3.48}
$$

completing the proof. □

Theorem 3.18 (Fractional Caffarelli-Kohn-Nirenberg inequality)**.** *Suppose that $Q \geq 2$, $s \in (0,1)$, $p > 1$, $\alpha \geq 1$, $\tau > 0$, $a \in (0,1]$, β_1, β_2, β, μ, $\gamma \in \mathbb{R}$, $\beta_1 + \beta_2 = \beta$ and*

$$\frac{1}{\tau}+\frac{\gamma}{Q}=a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right). \tag{3.49}$$

Suppose in addition that, $0 \leq \beta - \sigma$ with $\gamma = a\sigma + (1-a)\mu$, and

$$\beta-\sigma \leq s \text{ only if } \frac{1}{\tau}+\frac{\gamma}{Q}=\frac{1}{p}+\frac{\beta-s}{Q}. \tag{3.50}$$

Then for $u \in C_c^1(\mathbb{G})$ we have

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^a_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.51}$$

when $\frac{1}{\tau}+\frac{\gamma}{Q}>0$, and for $u \in C_c^1(\mathbb{G}\setminus\{e\})$ we have

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{G})} \leq C[u]^a_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})}, \tag{3.52}$$

when $\frac{1}{\tau}+\frac{\gamma}{Q}<0$. Here e is the identity element of $\mathbb{G}$.

Proof. Firstly, let us consider the case (3.50), that is, $\beta-\sigma\leq s$ and $\frac{1}{\tau}+\frac{\gamma}{Q}=\frac{1}{p}+\frac{\beta-s}{Q}$. By combining Lemma 3.17, $\lambda=2^k$, $r=1$, $R=2$ and $\Omega=A_k$, we obtain

$$\left(\fint_{A_{k,q}}|u-u_{A_{k,q}}|^\tau dx\right)^{\frac{1}{\tau}}\leq C2^{\frac{ak(sp-Q)}{p}}[u]^a_{s,p,q,A_{k,q}}\left(\fint_{A_{k,q}}|u|^\alpha dx\right)^{\frac{1-a}{\alpha}}, \tag{3.53}$$

where A_k is defined in (3.44) and $k\in\mathbb{Z}$. From (3.53) we obtain

$$\begin{aligned}\int_{A_k}|u|^\tau dx&=\int_{A_k}|u-u_{A_k}+u_{A_k}|^\tau dx\\&\leq C\left(\int_{A_k}|u_{A_k}|^\tau dx+\int_{A_k}|u-u_{A_k}|^\tau dx\right)\\&=C\left(\int_{A_k}|u_{A_k}|^\tau dx+\frac{|A_k|}{|A_k|}\int_{A_k}|u-u_{A_k}|^\tau dx\right)\\&=C\left(|A_k||u_{A_k}|^\tau+|A_k|\fint_{A_k}|u-u_{A_k}|^\tau dx\right)\\&\leq C\left(|A_k||u_{A_k}|^\tau+2^{\frac{ak(sp-Q)\tau}{p}}|A_k|[u]^{a\tau}_{s,p,A_k}\left(\frac{1}{|A_k|}\int_{A_k}|u|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}}\right)\\&\leq C\left(2^{Qk}|u_{A_k}|^\tau+2^{\frac{ak(sp-Q)\tau}{p}}2^{kQ}2^{-\frac{Q(1-a)\tau k}{\alpha}}[u]^{a\tau}_{s,p,A_k}\|u\|^{(1-a)\tau}_{L^\alpha(A_k)}\right).\end{aligned}\tag{3.54}$$

Then, from (3.54) we establish

$$\begin{aligned}&\int_{A_k}|x|^{\gamma\tau}|u|^\tau dx\leq 2^{(k+1)\gamma\tau}\int_{A_k}|u|^\tau dx\leq C2^{(Q+\gamma\tau)k}|u_{A_k}|^\tau\\&+C2^{\gamma\tau k}2^{kQ}2^{\frac{ak(sp-Q)\tau}{p}}2^{-\frac{Q(1-a)\tau k}{\alpha}}[u]^{a\tau}_{s,p,A_k}\|u\|^{(1-a)\tau}_{L^\alpha(A_k)}=C2^{(Q+\gamma\tau)k}|u_{A_k}|^\tau\\&+C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}\right)k}\left(\int_{A_k}\int_{A_k}\frac{2^{kp\beta_1}2^{kp\beta_2}|u(x)-u(y)|^p}{2^{kp\beta}|y^{-1}\,x|^{Q+sp}}dxdy\right)^{\frac{a\tau}{p}}\\&\times\left(\int_{A_k}\frac{2^{k\alpha\mu}}{2^{k\alpha\mu}}|u(x)|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}}\leq C2^{(Q+\gamma\tau)k}|u_{A_k}|^\tau\\&+C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\right)k}\left(\int_{A_k}\int_{A_k}\frac{|x|^{p\beta_1}|y|^{p\beta_2}|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\frac{a\tau}{p}}\\&\times\left(\int_{A_k}|x|^{\alpha\mu}|u(x)|^\alpha dx\right)^{\frac{(1-a)\tau}{\alpha}}\leq C2^{(Q+\gamma\tau)k}|u_{A_k}|^\tau\\&+C2^{\left(\gamma\tau+Q+\frac{a(sp-Q)\tau}{p}-\frac{Q(1-a)\tau}{\alpha}-a\beta\tau-\mu\tau(1-a)\right)k}[u]^{a\tau}_{s,p,\beta,A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_k)}.\end{aligned}$$

From (3.49), we have

$$
\begin{aligned}
&\gamma\tau + Q + \frac{a(sp-Q)\tau}{p} - \frac{Q(1-a)\tau}{\alpha} - a\beta\tau - \mu\tau(1-a) \\
&\quad = Q\tau\left(\frac{\gamma}{Q} + \frac{1}{\tau} + \frac{a(sp-Q)}{Qp} - \frac{(1-a)}{\alpha} - \frac{a\beta}{Q} - \frac{\mu(1-a)}{Q}\right) \\
&\quad = Q\tau\left(a\left(\frac{1}{p} + \frac{\beta-s}{Q}\right) + (1-a)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right) + \frac{a(sp-Q)}{Qp} - \frac{(1-a)}{\alpha} - \frac{a\beta}{Q} - \frac{\mu(1-a)}{Q}\right) \\
&\quad = 0.
\end{aligned}
$$

Thus, we obtain

$$
\int_{A_k} |x|^{\gamma\tau}|u|^\tau dx \leq C2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau + C[u]^{a\tau}_{s,p,\beta,A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_k)}, \tag{3.55}
$$

and by summing over k from m to n, we get

$$
\begin{aligned}
\int_{\cup_{k=m}^n A_k} |x|^{\gamma\tau}|u|^\tau dx = \int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx &\leq C\sum_{k=m}^n 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \\
&+ C\sum_{k=m}^n [u]^{a\tau}_{s,p,\beta,A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_k)},
\end{aligned} \tag{3.56}
$$

where $k, m, n \in \mathbb{Z}$ and $m \leq n-2$.

Let us show (3.51). By choosing n such that

$$
\operatorname{supp} u \subset B_{2^n}, \tag{3.57}
$$

where B_{2^n} is a quasi-ball of $\mathbb{G}$ with the radius 2^n.

The following known inequality will be used in the proof.

Lemma 3.19 (Lemma 2.2, [65]). *Let $\xi > 1$ and $\eta > 1$. Then exists a positive constant C depending ξ and η such that $1 < \zeta < \xi$,*

$$
(|a| + |b|)^\eta \leq \zeta|a|^\eta + \frac{C}{(\zeta-1)^{\eta-1}}|b|^\eta, \quad \forall\, a, b \in \mathbb{R}. \tag{3.58}
$$

Let us consider the following integral

$$
\begin{aligned}
&\fint_{A_{k+1,q}\cup A_{k,q}} \left|u - \fint_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx \\
&\qquad = \frac{1}{|A_{k+1,q}| + |A_{k,q}|}\int_{A_{k+1,q}\cup A_{k,q}} \left|u - \fint_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx \\
&= \frac{1}{|A_{k+1,q}| + |A_{k,q}|}\left(\int_{A_{k+1,q}} \left|u - \fint_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx + \int_{A_{k,q}} \left|u - \fint_{A_{k+1,q}\cup A_{k,q}} u\right|^\tau dx\right).
\end{aligned}
$$

Then, we compute

$$
\begin{aligned}
&\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&=\frac{1}{|A_{k+1}|+|A_k|}(\int_{A_{k+1}}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&+\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx)\\
&\geq\frac{1}{|A_{k+1}|+|A_k|}\int_{A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx\\
&\geq\frac{1}{|A_{k+1}|+|A_k|}\left|\int_{A_k}\left(u-\fint_{A_{k+1}\cup A_k}u\right)dx\right|^{\tau}\\
&=\frac{1}{|A_{k+1}|+|A_k|}|\int_{A_k}udx-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_k}udx\\
&-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx|^{\tau}\\
&=\frac{1}{|A_{k+1}|+|A_k|}|\frac{|A_{k+1}|}{|A_{k+1}|+|A_k|}\int_{A_k}udx\\
&-\frac{|A_k|}{|A_{k+1}|+|A_k|}\int_{A_{k+1}}udx|^{\tau}\\
&=\frac{1}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left||A_{k+1}|\int_{A_k}udx-|A_k|\int_{A_{k+1}}udx\right|^{\tau}\\
&=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}\left|\frac{1}{|A_k|}\int_{A_k}udx-\frac{1}{|A_{k+1}|}\int_{A_{k+1}}udx\right|^{\tau}\\
&=\frac{|A_{k+1}|^{\tau}|A_k|^{\tau}}{(|A_{k+1}|+|A_k|)^{\tau+1}}|u_{A_{k+1}}-u_{A_k}|^{\tau}\\
&\geq C|u_{A_{k+1}}-u_{A_k}|^{\tau}.
\end{aligned}
\tag{3.59}
$$

By combining (3.59) and Lemma 3.17, we get

$$
|u_{A_{k+1}}-u_{A_k}|^{\tau}\leq C\fint_{A_{k+1}\cup A_k}\left|u-\fint_{A_{k+1}\cup A_k}u\right|^{\tau}dx
$$

$$
\leq C2^{\frac{ak(sp-Q)}{p}}[u]^{\tau a}_{s,p,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{(1-a)\tau}{\alpha}}.
$$

By using this fact with $\tau = 1$, we get

$$|u_{A_k}| \leq |u_{A_{k+1}} - u_{A_k}| + |u_{A_{k+1}}|$$
$$\leq |u_{A_{k+1}}| + C2^{\frac{ak(sp-Q)}{p}} [u]^a_{s,p,A_{k+1}\cup A_k} \left(⨍_{A_{k+1}\cup A_k} |u|^\alpha dx \right)^{\frac{(1-a)}{\alpha}}, \quad (3.60)$$

and from Lemma 3.19 and $\eta = \tau$, $\zeta = 2^{\gamma\tau+Q}c$, where $c = \frac{2}{1+2^{\gamma\tau+Q}} < 1$, since $\gamma\tau + Q > 0$, we obtain

$$2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \leq c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1}}|^\tau + C[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}.$$

By summing over k from m to n and by using (3.57) we have

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \leq \sum_{k=m}^{n} c2^{(k+1)(\gamma\tau+Q)}|u_{A_{k+1}}|^\tau$$
$$+ C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu(x)u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \quad (3.61)$$

From (3.61), we get

$$(1-c)\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \leq 2^{(\gamma\tau+Q)m}|u_{A_m}|^\tau + (1-c)\sum_{k=m+1}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau$$
$$\leq C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \quad (3.62)$$

This yields

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \leq C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \quad (3.63)$$

By using (3.56) and (3.63), we have

$$\int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \leq C\sum_{k=m}^{n}[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \quad (3.64)$$

Assume $s, t \geq 0$ be such that $s + t \geq 1$. Then for any $x_k, y_k \geq 0$, we have

$$\sum_{k=m}^{n} x_k^s y_k^t \leq \left(\sum_{k=m}^{n} x_k\right)^s \left(\sum_{k=m}^{n} y_k\right)^t. \quad (3.65)$$

By using this inequality in (3.64) with $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, $\frac{a}{p} + \frac{1-a}{\alpha} \geq \frac{1}{\tau}$ and $s \geq \beta - \sigma$, we obtain

$$\int_{\{|x|>2^m\}} |x|^{\gamma\tau}|u|^\tau dx \leq C[u]^{a\tau}_{s,p,\beta,\cup_{k=m}^\infty A_k}\||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=m}^\infty A_k)}. \quad (3.66)$$

Inequality (3.51) is proved.

Let us show (3.52). The strategy of the proof is similar to the previous case. By choosing m such that

$$\operatorname{supp} u \cap B_{2^m} = \emptyset. \quad (3.67)$$

By using Lemma 3.17 we get

$$|u_{A_{k+1}} - u_{A_k}|^\tau \leq C2^{\frac{a\tau k(sp-Q)}{p}} [u]^{\tau a}_{s,p,A_{k+1}\cup A_k} \left(\fint_{A_{k+1}\cup A_k} |u|^\alpha dx \right)^{\frac{(1-a)\tau}{\alpha}}.$$

From Lemma 3.19 and choosing $c = \frac{1+2^{\gamma\tau+Q}}{2} < 1$, since $\gamma\tau + Q < 0$, we establish

$$2^{(\gamma\tau+Q)(k+1)}|u_{A_{k+1}}|^\tau \leq c2^{k(\gamma\tau+Q)}|u_{A_k}|^\tau + C[u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)},$$

and by summing over k from m to n and by using (3.67) we obtain

$$\sum_{k=m}^{n} 2^{(\gamma\tau+Q)k}|u_{A_k}|^\tau \leq C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.68}$$

By using (3.56) and (3.68), we obtain that

$$\int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \leq C \sum_{k=m-1}^{n-1} [u]^{\tau a}_{s,p,\beta,A_{k+1}\cup A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(A_{k+1}\cup A_k)}. \tag{3.69}$$

From (3.65) we get

$$\int_{\{|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau dx \leq C[u]^{\tau a}_{s,p,\beta,\cup_{k=-\infty}^{n} A_k} \||x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=-\infty}^{n} A_k)}. \tag{3.70}$$

The proof of the case $s \geq \beta - \sigma$ is complete.

Let us prove the case of $\beta - \sigma > s$. Without loss of generality, we suppose that

$$[u]_{s,p,\beta} = \|u\|_{L^\alpha(\mathbb{G})} = 1, \tag{3.71}$$

where

$$\frac{1}{p} + \frac{\beta - s}{Q} \neq \frac{1}{\alpha} + \frac{\mu}{Q}.$$

We also suppose that $a_1 > 0$, $1 > a_2$ and τ_1, $\tau_2 > 0$ with

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1-a_2}{\alpha}, \tag{3.72}$$

and

$$\begin{aligned} &\text{if} \quad \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0, \quad \text{then} \quad \frac{1}{\tau_1} = \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}, \\ &\text{if} \quad \frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} \leq 0, \quad \text{then} \quad \frac{1}{\tau} > \frac{1}{\tau_1} \geq \frac{a_1}{p} + \frac{1-a_1}{\alpha} - \frac{a_1 s}{Q}. \end{aligned} \tag{3.73}$$

By taking $\gamma_1 = a_1\beta + (1-a_1)\mu$ and $\gamma_2 = a_2(\beta - s) + (1-a_2)\mu$, we have

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} \geq a_1\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_1)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right) \tag{3.74}$$

and

$$\frac{1}{\tau_2} + \frac{\gamma_2}{Q} = a_2\left(\frac{1}{p} + \frac{\beta - s}{Q}\right) + (1-a_2)\left(\frac{1}{\alpha} + \frac{\mu}{Q}\right). \tag{3.75}$$

Assume a_1 and a_2 be such that

$$|a - a_1| \text{ and } |a - a_2| \text{ are small enough}, \tag{3.76}$$

$$a_2 < a < a_1, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} > \frac{1}{\alpha} + \frac{\mu}{Q}, \tag{3.77}$$

$$a_1 < a < a_2, \text{ if } \frac{1}{p} + \frac{\beta - s}{Q} < \frac{1}{\alpha} + \frac{\mu}{Q}. \tag{3.78}$$

By combining (3.76)-(3.78) in (3.74), (3.75) and (3.49), we get

$$\frac{1}{\tau_1} + \frac{\gamma_1}{Q} > \frac{1}{\tau} + \frac{\gamma}{Q} > \frac{1}{\tau_2} + \frac{\gamma_2}{Q} > 0. \tag{3.79}$$

From (3.73) in the case $\frac{a}{p} + \frac{1-a}{\alpha} - \frac{as}{Q} > 0$ with $a > 0$, $\beta - \sigma > s$ and (3.76), we get

$$\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1)\left(\frac{1}{p} - \frac{s}{Q} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma) > 0, \tag{3.80}$$

and

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\left(\frac{1}{p} - \frac{1}{\alpha}\right) + \frac{a}{Q}(\beta - \sigma - s) > 0. \tag{3.81}$$

By combining (3.73), (3.80) and (3.81), we get

$$\tau_1 > \tau, \quad \tau_2 > \tau.$$

Thus, by using last fact, (3.76) and Hölder's inequality, we get

$$\||x|^{\gamma}u\|_{L^{\tau}(\mathbb{G}\setminus B_1)} \leq C\||x|^{\gamma_1}u\|_{L^{\tau_1}(\mathbb{G})}, \tag{3.82}$$

and

$$\||x|^{\gamma}u\|_{L^{\tau}(B_1)} \leq C\||x|^{\gamma_2}u\|_{L^{\tau_2}(\mathbb{G})}, \tag{3.83}$$

where B_1 is the unit quasi-ball. By using the previous case, we get

$$\||x|^{\gamma_1}u\|_{L^{\tau_1}(\mathbb{G})} \leq C[u]^{a_1}_{s,p,\beta}\||x|^{\mu}u\|^{1-a_1}_{L^{\alpha}(\mathbb{G})} \leq C, \tag{3.84}$$

and

$$\||x|^{\gamma_2}u\|_{L^{\tau_2}(\mathbb{G})} \leq C[u]^{a_2}_{s,p,\beta}\||x|^{\mu}u\|^{1-a_2}_{L^{\alpha}(\mathbb{G})} \leq C. \tag{3.85}$$

The proof of Theorem 3.18 is complete. □

Remark 3.20. *In the Abelian case $(\mathbb{R}^N, +)$ with the standard Euclidean distance instead of quasi-norm in Theorem 3.18, we get the (Euclidean) fractional Caffarelli-Kohn-Nirenberg inequality (see, e.g. [41], Theorem 1.1).*

Remark 3.21. *In the Abelian case $(\mathbb{R}^N, +)$ with the standard Eucledian distance instead of the quasi-norm and $s \to 1^-$ in (3.52), we get classical Caffarelli-Kohn-Nirenberg inequality.*

Remark 3.22. *By taking in (3.52) $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, we get an analogue of the fractional Hardy inequality on homogeneous Lie groups (Theorem 3.4).*

Remark 3.23. *In the Abelian case $(\mathbb{R}^N, +)$ with the standard Eucledian distance instead of the quasi-norm and by taking in (3.52) $a = 1$, $\tau = p$, $\beta_1 = \beta_2 = 0$, and $\gamma = -s$, we get the fractional Hardy inequality (Theorem 1.1, [1]).*

Remark 3.24. *By taking in (3.51) $a = 1$, $\tau = p^*$, $\beta_1 = \beta_2 = 0$, and $\gamma = 0$, we get an analogue of the fractional Sobolev inequality on homogeneous Lie groups (Theorem 3.9).*

Now we consider the critical case $\frac{1}{\tau}+\frac{\gamma}{Q}=0$.

Theorem 3.25 (Fractional critical Caffarelli-Kohn-Nirenberg inequality)**.** *Suppose that $Q \geq 2$, $s \in (0,1)$, $p>1$, $\alpha \geq 1$, $\tau>1$, $a \in (0,1]$, β_1, β_2, β, μ, $\gamma \in \mathbb{R}$, $\beta_1+\beta_2=\beta$,*

$$\frac{1}{\tau}+\frac{\gamma}{Q}=a\left(\frac{1}{p}+\frac{\beta-s}{Q}\right)+(1-a)\left(\frac{1}{\alpha}+\frac{\mu}{Q}\right). \tag{3.86}$$

Suppose in addition that, $0 \leq \beta-\sigma \leq s$ with $\gamma=a\sigma+(1-a)\mu$.

If $\frac{1}{\tau}+\frac{\gamma}{Q}=0$ and $\operatorname{supp} u \subset B_R$, then, we have

$$\left\|\frac{|x|^{\gamma}}{\ln\frac{2R}{|x|}}u\right\|_{L^{\tau}(\mathbb{G})} \leq C[u]^{a}_{s,p,\beta}\||x|^{\mu}u\|^{1-a}_{L^{\alpha}(\mathbb{G})},\ u \in C^1_c(\mathbb{G}), \tag{3.87}$$

where $B_R=\{x \in \mathbb{G}: |x|<R\}$ is the quasi-ball and $0<r<R$.

Proof of Theorem 3.25. The proof is similar to the proof of Theorem 3.18. In (3.55), by summarising over k from m to n and by fixing $\varepsilon>0$, we get

$$\int_{\{|x|>2^m\}}\frac{|x|^{\gamma\tau}}{\ln^{1+\varepsilon}\left(\frac{2R}{|x|}\right)}|u|^{\tau}dx \leq C\sum_{k=m}^{n}\frac{1}{(n+1-k)^{1+\varepsilon}}|u_{A_k}|^{\tau}$$
$$+C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_k}\||x|^{\mu}(x)u\|^{(1-a)\tau}_{L^{\alpha}(A_k)}. \tag{3.88}$$

By using Lemma 3.17, we get

$$|u_{A_{k+1}}-u_{A_k}| \leq C2^{\frac{ak(sp-Q)}{p}}[u]^{a}_{s,p,A_{k+1}\cup A_k}\left(\fint_{A_{k+1}\cup A_k}|u|^{\alpha}dx\right)^{\frac{1-a}{\alpha}}.$$

From Lemma 3.19 with $\zeta=\frac{(n+1-k)^{\varepsilon}}{(n+\frac{1}{2}-k)^{\varepsilon}}$ we establish

$$\frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\varepsilon}} \leq \frac{|u_{A_{k+1}}|^{\tau}}{(n+\frac{1}{2}-k)^{\varepsilon}}$$
$$+C(n+1-k)^{\tau-1-\varepsilon}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{3.89}$$

For $\varepsilon>0$ and $n \geq k$, we have

$$\frac{1}{(n-k+1)^{\varepsilon}}-\frac{1}{(n-k+\frac{3}{2})^{\varepsilon}} \sim \frac{1}{(n-k+1)^{1+\varepsilon}}. \tag{3.90}$$

By combining this fact, (3.89), (3.90) and $\varepsilon=\tau-1$, we get

$$\sum_{k=m}^{n}\frac{|u_{A_k}|^{\tau}}{(n+1-k)^{\tau}} \leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{3.91}$$

By using (3.88) and (3.91), we have

$$\int_{\{|x|>2^m\}}\frac{|x|^{\gamma\tau}}{\ln^{\tau}\frac{2R}{|x|}}|u|^{\tau}dx \leq C\sum_{k=m}^{n}[u]^{a\tau}_{s,p,\beta,A_{k+1}\cup A_k}\||x|^{\mu}u\|^{(1-a)\tau}_{L^{\alpha}(A_{k+1}\cup A_k)}. \tag{3.92}$$

By combining (3.65) with (3.86) and $0 \le \beta - \sigma \le s$, where $s = \frac{\tau a}{p}$, $t = \frac{(1-a)\tau}{\alpha}$, we have $s + t \ge 1$ and we get

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^\tau \frac{2R}{|x|}} |u|^\tau dx \le C \sum_{k=m}^{n} [u]^{a\tau}_{s,p,\beta,\cup_{k=m}^{\infty} A_k} \| |x|^\mu u\|^{(1-a)\tau}_{L^\alpha(\cup_{k=m}^{\infty} A_k)}, \tag{3.93}$$

completing the proof. □

3.6. **Fractional Logarithmic inequalities.** In this section, we show fractional logarithmic inequalities on homogeneous Lie group. By the way, we need some preliminary results. Firstly, we show weighted Hölder's inequality on $\mathbb{G}$.

Lemma 3.26. *Assume that $1 < p \le r \le q \le \infty$, $a \in [0,1]$, $\alpha \in \mathbb{R}$, $|x|^\alpha u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$ with*

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}, \tag{3.94}$$

then we have

$$\| |x|^\alpha u\|_{L^r(\mathbb{G})} \le \| |x|^\alpha u\|^a_{L^p(\mathbb{G})} \| |x|^\alpha u\|^{1-a}_{L^q(\mathbb{G})}. \tag{3.95}$$

Proof. By using Hölder's inequality we obtain

$$\begin{aligned} \| |x|^\alpha u\|^r_{L^r(\mathbb{G})} &= \int_{\mathbb{G}} |x|^{\alpha r} |u(x)|^r dx = \int_{\mathbb{G}} (|x|^\alpha |u(x)|)^{ar} (|x|^\alpha |u(x)|)^{(1-a)r} dx \\ &\le \left(\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^p dx \right)^{\frac{ar}{p}} \left(\int_{\mathbb{G}} |x|^{\alpha q} |u(x)|^q dx \right)^{\frac{(1-a)r}{q}} \\ &= \| |x|^\alpha u\|^{ar}_{L^p(\mathbb{G})} \| |x|^\alpha u\|^{(1-a)r}_{L^q(\mathbb{G})}, \end{aligned} \tag{3.96}$$

with

$$\frac{ar}{p} + \frac{(1-a)r}{q} = 1. \tag{3.97}$$

□

Now let us show logarithmic Hölder's inequality.

Lemma 3.27 (Logarithmic Hölder's inequality)**.** *Suppose that $|x|^\alpha u \in L^p(\mathbb{G}) \cap L^q(\mathbb{G})$ with some $\alpha \in \mathbb{R}$, $1 < p < q \le \infty$. Then we have*

$$\int_{\mathbb{G}} \frac{(|x|^{\alpha p} |u|^p)}{\| |x|^\alpha u\|^p_{L^p(\mathbb{G})}} \log \left(\frac{|x|^{\alpha p} |u|^p}{\| |x|^\alpha u\|^p_{L^p(\mathbb{G})}} \right) dx \le \frac{q}{q-p} \log \left(\frac{\| |x|^\alpha u\|^p_{L^q(\mathbb{G})}}{\| |x|^\alpha u\|^p_{L^p(\mathbb{G})}} \right). \tag{3.98}$$

Proof. Let us consider the following function

$$F\left(\frac{1}{r}\right) = \log \left(\| |x|^\alpha u\|_{L^r(\mathbb{G})} \right). \tag{3.99}$$

Firstly, we need to prove the function (3.99) is convex. By using Lemma 3.26, we obtain

$$\begin{aligned} F\left(\frac{1}{r}\right) &= \log \left(\| |x|^\alpha u\|_{L^r(\mathbb{G})} \right) \le \log \left(\| |x|^\alpha u\|^a_{L^p(\mathbb{G})} \| |x|^\alpha u\|^{1-a}_{L^q(\mathbb{G})} \right) \\ &= \log \left(\| |x|^\alpha u\|^a_{L^p(\mathbb{G})} \right) + \log \left(\| |x|^\alpha u\|^{1-a}_{L^q(\mathbb{G})} \right) = aF\left(\frac{1}{p}\right) + (1-a)F\left(\frac{1}{q}\right), \end{aligned} \tag{3.100}$$

with $a \in [0,1]$ and $\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q}$.

Since we have

$$F(r) = r \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx, \tag{3.101}$$

the derivative of (3.101) is

$$\begin{aligned} F'(r) &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx + r \left(\log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx \right)'_r \\ &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx + r \frac{\left(\int_{\mathbb{G}} (|x|^{\alpha} u(x))^{\frac{1}{r}} dx \right)'_r}{\int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx} \\ &= \log \int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx - \frac{1}{r} \frac{\int_{\mathbb{G}} (|x|^{\alpha} u(x))^{\frac{1}{r}} \log(|x|^{\alpha} |u(x)|) dx}{\int_{\mathbb{G}} |x|^{\frac{\alpha}{r}} |u(x)|^{\frac{1}{r}} dx}. \end{aligned} \tag{3.102}$$

From (3.100) $F(r)$ is convex, hence, we get

$$F'(r) \geq \frac{F(r') - F(r)}{r' - r}, \quad r' > r > 0. \tag{3.103}$$

With $r = \frac{1}{p}$ and $r' = \frac{1}{q}$ it yields

$$\begin{aligned} p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^{p} \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} &- \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx \\ &\leq \frac{qp}{q-p} \log \left(\int_{\mathbb{G}} \frac{\||x|^{\alpha} u\|_{L^{q}(\mathbb{G})}}{\||x|^{\alpha} u\|_{L^{p}(\mathbb{G})}} \right). \end{aligned} \tag{3.104}$$

We have

$$\begin{aligned} \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx &= \frac{\log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} \\ &= \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} \log \||x|^{\alpha} u\|^{p}_{L^{p}(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx}. \end{aligned} \tag{3.105}$$

By using last fact in (3.104) we establish logarithmic Hölder's inequality

$$\begin{aligned} p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^{p} \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} &- \log \int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx \\ &= p \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^{p} \log |x|^{\alpha} |u| dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} \log \||x|^{\alpha} u\|^{p}_{L^{p}(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} \\ &= \frac{\int_{\mathbb{G}} ||x|^{\alpha} u|^{p} \log |x|^{\alpha p} |u|^{p} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} - \frac{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} \log \||x|^{\alpha} u\|^{p}_{L^{p}(\mathbb{G})} dx}{\int_{\mathbb{G}} |x|^{\alpha p} |u(x)|^{p} dx} \\ &= \int_{\mathbb{G}} \frac{(|x|^{\alpha p} |u|^{p})}{\||x|^{\alpha} |u|\|^{p}_{L^{p}(\mathbb{G})}} \log \left(\frac{|x|^{\alpha p} |u|^{p}}{\||x|^{\alpha} u\|^{p}_{L^{p}(\mathbb{G})}} \right) \leq \frac{q}{q-p} \log \left(\frac{\||x|^{\alpha} u\|^{p}_{L^{q}(\mathbb{G})}}{\||x|^{\alpha} u\|^{p}_{L^{p}(\mathbb{G})}} \right). \end{aligned} \tag{3.106}$$

□

3.6.1. *Fractional Logarithmic Sobolev inequality.* In this subsection, we present the fractional logarithmic Sobolev inequality on $\mathbb{G}$.

Theorem 3.28 (Fractional Logarithmic Sobolev inequality). *Let $p > 1$, $s \in (0,1)$, $Q > sp$ be such that $p^* = p^*(Q,s,p) = \frac{Qp}{Q-sp}$. Let $|\cdot|$ be a quasi-norm on $\mathbb{G}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$, we have the fractional logarithmic Sobolev's inequality*

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{Q}{sp} \log\left(C\frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}}\right), \tag{3.107}$$

where C is a positive constant independent on u.

Proof. By using weighted logarithmic Hölder's inequality (3.98) with $\alpha = 0$, we obtain

$$\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{q}{q-p} \log\left(\frac{\|u\|^p_{L^q(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right). \tag{3.108}$$

By the assumption we have $1 \leq p < q = p^* = \frac{pQ}{Q-sp}$ and by using Theorem 3.9, we get

$$\begin{aligned}\int_{\mathbb{G}} \frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|u|^p}{\|u\|^p_{L^p(\mathbb{G})}}\right) dx &\leq \frac{p^*}{p^*-p} \log\left(\frac{\|u\|^p_{L^{p^*}(\mathbb{G})}}{\|u\|^p_{L^p(\mathbb{G})}}\right)\\ &\overset{(3.31)}{\leq} \frac{p^*}{p^*-p} \log\left(C\frac{[u]^p_{s,p}}{\|u\|^p_{L^p(\mathbb{G})}}\right).\end{aligned} \tag{3.109}$$

Here we have

$$\frac{p^*}{p^*-p} = \frac{\frac{pQ}{Q-sp}}{\frac{pQ}{Q-sp}-p} = \frac{\frac{Q}{Q-sp}}{\frac{Q}{Q-sp}-1} = \frac{Q}{sp}.$$

$\square$

Remark 3.29. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ and from (3.107) we get the logarithmic Sobolev inequality from* [36].

3.6.2. *Fractional Logarithmic Hardy-Sobolev type inequality.* Motivated by the above result, in this section we prove the fractional logarithmic Hardy-Sobolev inequality on the homogeneous Lie groups.

Theorem 3.30. *Suppose that $p > 1$, $s \in (0,1)$, $Q > 2$, $0 < \beta < sp$ and $Q > sp$ be such that $p^*_{s,\beta} = \frac{p(Q-\beta)}{Q-sp}$. Then for any $u \in W^{s,p}(\mathbb{G})$ and for any quasi-norm $|\cdot|$ of $\mathbb{G}$,* *the fractional logarithmic Hardy-Sobolev's type inequality:*

$$\int_{\mathbb{G}} \frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) dx \leq \frac{Q-\beta}{sp-\beta}\log\left(C\frac{[u]^p_{s,p}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right), \tag{3.110}$$

where C is a positive constant and independent of u.

Proof. In the assumptions of Lemma 3.27, by taking $\alpha = -\frac{\beta p}{p^*_{s,\beta}}$.Then, it is easy to see that $p < p^*_{s,\beta} = q$. Hence by using Lemma 3.27 and Theorem 3.10 with $\alpha = -\frac{\beta p}{p^*_{s,\beta}}$, we get

$$\begin{aligned}\int_{\mathbb{G}} \frac{(|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p)}{\||x|^{-\frac{\beta p}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}} \log\left(\frac{|x|^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|^{-\frac{\beta p}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) dx &\overset{(3.98)}{\leq} \frac{p^*_{s,\beta}}{p^*_{s,\beta}-p}\log\left(\frac{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^{p^*_{s,\beta}}(\mathbb{G})}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right) \\ &\overset{(3.37)}{\leq} \frac{p^*_{s,\beta}}{p^*_{s,\beta}-p}\log\left(C\frac{[u]^p_{s,p}}{\||x|^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{G})}}\right).\end{aligned} \tag{3.111}$$

Finally, we compute

$$\frac{p^*_{s,\beta}}{p^*_{s,\beta}-p} = \frac{\frac{p(Q-\beta)}{Q-sp}}{\frac{p(Q-\beta)}{Q-sp}-p} = \frac{Q-\beta}{sp-\beta},$$

with $sp > \beta > 0$. ☐

Remark 3.31. *In (3.110) with $\beta = 0$, we have the fractional logarithmic Sobolev inequality on $\mathbb{G}$. However, from (3.110) it does not follow the fractional logarithmic Hardy inequality since in Lemma 3.27 we have the assumtion $p < q = p^*_{s,\beta}$. To get the fractional Hardy inequality we have to set $\beta = sp$, then $p = q = p^*_{s,sp}$.*

Remark 3.32. *In the Abelian case $(\mathbb{R}^N, +)$, $Q = N$ with $|\cdot| = |\cdot|_E$ where $|\cdot|_E$ is the standard Euclidean distance, combining (3.110) and (3.37) we obtain the following* *fractional logarithmic Hardy-Sobolev inequality:*

$$\int_{\mathbb{R}^N} \frac{|x|_E^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|x|_E^{-\frac{\beta p}{p^*_{s,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right) dx$$
$$\leq \frac{N-\beta}{sp-\beta}\log\left(C\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|_E^{N+sp}}dxdy}{\||x|_E^{-\frac{\beta}{p^*_{s,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right), \quad (3.112)$$

for all $u \in W^{s,p}(\mathbb{R}^N)$.

Remark 3.33. *In the Abelian case $(\mathbb{R}^N,+)$, $Q = N$ with $|\cdot| = |\cdot|_E$ where $|\cdot|_E$ is the standard Euclidean distance and $s \to 1^-$, combining (3.110) and (3.37) we have the following fractional logarithmic Hardy-Sobolev inequality:*

$$\int_{\mathbb{R}^N} \frac{|x|_E^{-\frac{\beta p}{p^*_{1,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}} \log\left(\frac{|x|_E^{-\frac{\beta p}{p^*_{1,\beta}}}|u|^p}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right) dx$$
$$\leq \frac{N-\beta}{p-\beta}\log\left(C\frac{\|\nabla u\|^p_{L^p(\mathbb{R}^N)}}{\||x|_E^{-\frac{\beta}{p^*_{1,\beta}}}u\|^p_{L^p(\mathbb{R}^N)}}\right), \quad (3.113)$$

and also, setting $\beta = 0$, we get result from [36].

3.6.3. *Fractional Logarithmic Gagliardo-Nirenberg inequality.* In this subsection, we show fractional logarithmic Gagliardo-Nirenberg inequality on $\mathbb{G}$.

Theorem 3.34 (Fractional Logarithmic Gagliardo-Nirenberg inequality)**.** *Under the assumptions of Theorem 3.14 with the parameters $1 \leq p < \infty$, $1 < q < \infty$ and $q \leq p^*$, there exists $C = C(Q,p,s,q) > 0$ such that for all measurable and compactly supported u we have*

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right)dx \leq \frac{1}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right)dx. \quad (3.114)$$

Proof. From the fractional Gagliardo-Nirenberg inequality (3.40) and the logarithmic Hölder inequality (3.98), we get

$$\int_{\mathbb{G}} \frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\log\left(\frac{|u|^q}{\|u\|^q_{L^q(\mathbb{G})}}\right)dx \leq \frac{1}{1-\frac{q}{\tau}}\log\left(\frac{\|u\|^q_{L^\tau(\mathbb{G})}}{\|u\|^q_q}\right)$$
$$\leq \frac{1}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^{qa}_{s,p}\|u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\|u\|^q_{L^q(\mathbb{G})}}\right) = \frac{a}{1-\frac{q}{\tau}}\log\left(C\frac{[u]^q_{s,p}}{\|u\|^q_{L^q(\mathbb{G})}}\right). \quad (3.115)$$

$\square$

Remark 3.35. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ and from (3.114) we get the logarithmic Sobolev inequality in* [36].

3.6.4. *Fractional Logarithmic Caffarelli-Kohn-Nirenberg inequality.* Now we present the fractional logarithmic CKN type inequality on homogeneous groups.

Theorem 3.36 (Fractional Logarithmic CKN inequality)**.** *Under the assumptions of Theorem 3.18 with*

$$\alpha = \beta = \mu, \;\; 1 < q < p^*, \;\; 1 < p < Q, \;\; \beta p + Q > 0, \;\; \beta q + Q > 0, \tag{3.116}$$

there exists a positive constant C such that

$$\int_{\mathbb{G}} \frac{(|x|^{\alpha}|u|)^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right) dx \leq \frac{1}{1-\frac{q}{p^*}} \log\left(\frac{[u]_{s,p,\alpha}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right), \tag{3.117}$$

for all measurable and compactly supported u.

Proof. By taking $\alpha = \beta = \gamma$ in the assumptions of Theorem 3.18, we obtain that

$$\frac{1}{\tau} = \frac{a}{p^*} + \frac{1-a}{q}. \tag{3.118}$$

From the last fact with $q < p^*$ we have $q < \tau$. By combining these facts with weighted logarithmic Hölder's inequality and $\alpha = \beta = \gamma$ we get

$$\begin{aligned}
\int_{\mathbb{G}} \frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}} \log\left(\frac{|x|^{\alpha q}|u|^q}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right) dx &\leq \frac{\tau}{\tau-q} \log\left(\frac{\||x|^{\alpha}u\|^q_{L^{\tau}(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right) \\
&\leq \frac{\tau}{\tau-q} \log\left(C^a \frac{[u]^{aq}_{s,p,\alpha}\||x|^{\alpha}u\|^{(1-a)q}_{L^q(\mathbb{G})}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right) \\
&= \frac{a\tau}{\tau-q} \log\left(C \frac{[u]^q_{s,p,\alpha}}{\||x|^{\alpha}u\|^q_{L^q(\mathbb{G})}}\right).
\end{aligned} \tag{3.119}$$

Since $\alpha = \beta = \gamma$, we have

$$\frac{a\tau}{\tau-q} = \frac{p^*}{p^*-q}. \tag{3.120}$$

□

3.7. **Hardy-Littlewood-Sobolev inequality.** In this section, we show Hardy-Littlewood-Sobolev inequality. We prove this inequality by using Marcinkiewicz interpolation theorem.

Let us consider the integral operator

$$I_{\lambda,|\cdot|}u(x) = \int_{\mathbb{G}} \frac{u(y)}{|y^{-1}x|^{\lambda}} dy, \;\; 0 < \lambda < Q. \tag{3.121}$$

Note that when $Q > \alpha > 0$ and $\lambda = Q - \alpha$ we get the Riesz potential $I_{\lambda,|\cdot|} = I_{Q-\alpha,|\cdot|}$. First we give a short proof of a version of the Hardy-Littlewood-Sobolev inequality on $\mathbb{G}$.

Theorem 3.37. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < q < \infty$, $0 < \lambda < Q$, $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1$, and $u \in L^p(\mathbb{G})$. Then we have*

$$\|I_{\lambda,|\cdot|}u\|_{L^q(\mathbb{G})} \leq C\|u\|_{L^p(\mathbb{G})}, \tag{3.122}$$

where C is a positive constant independent of u.

Proof. As in the Euclidean case we will show that there is a constant $C > 0$, such that

$$m\{x : |K * u(x)| > \zeta\} \leq C\frac{\|u\|^q_{L^p(\mathbb{G})}}{\zeta^q}, \tag{3.123}$$

where m is the Haar measure on $\mathbb{G}$, $K(x) = |x|^{-\lambda}$ and $I_{\lambda,|\cdot|}u(x) = K * u(x)$, where $*$ is convolution. By using the Marcinkiewicz interpolation theorem we will prove (3.122). Let $K(x) = K_1(x) + K_2(x)$, where

$$K_1(x) := \begin{cases} K(x), & \text{if } |x| \leq \mu, \\ 0, & \text{if } |x| > \mu, \end{cases} \quad \text{and} \quad K_2(x) := \begin{cases} K(x), & \text{if } |x| > \mu, \\ 0, & \text{if } |x| \leq \mu, \end{cases} \tag{3.124}$$

$\mu > 0$. Then, we have $I_{\lambda,|\cdot|}u(x) = K * u(x) = K_1 * u(x) + K_2 * u(x)$, so

$$m\{x : |K * u(x)| > 2\zeta\} \leq m\{x : |K_1 * u(x)| > \zeta\} + m\{x : |K_2 * u(x)| > \zeta\}. \tag{3.125}$$

Therefore, it is enough to prove inequality (3.123) with 2ζ instead of ζ in the left-hand side of the inequality. Without loss of generality we can assume $\|u\|_{L^p(\mathbb{G})} = 1$ and by using Chebychev's and Minkowski's inequalities, we get

$$\begin{aligned} m\{x : |K_1 * u(x)| > \zeta\} &\leq \frac{\int_{|K_1 * u| > \zeta} |K_1 * u|^p dx}{\zeta^p} \\ &\leq \frac{\|K_1 * u\|^p_{L^p(\mathbb{G})}}{\zeta^p} \leq \frac{\|K_1\|^p_{L^1(\mathbb{G})}\|u\|^p_{L^p(\mathbb{G})}}{\zeta^p} = \frac{\|K_1\|^p_{L^1(\mathbb{G})}}{\zeta^p}. \end{aligned} \tag{3.126}$$

By combining (2.11) and (3.124), we have

$$\begin{aligned} \|K_1\|_{L^1(\mathbb{G})} &= \int_{0<|x|\leq\mu} |x|^{-\lambda} dx = \int_0^\mu r^{Q-1} r^{-\lambda} dr \int_{\mathfrak{S}} |y|^{-\lambda} d\sigma(y) \\ &= |\mathfrak{S}| \int_0^\mu r^{Q-\lambda-1} dr = \frac{|\mathfrak{S}|}{Q-\lambda}(r^{Q-\lambda}|_0^\mu) = \frac{|\mathfrak{S}|}{Q-\lambda}\mu^{Q-\lambda}, \end{aligned} \tag{3.127}$$

where $|\mathfrak{S}|$ is the dimensional surface measure of the unit quasi-sphere $\mathfrak{S}$. By using last fact in (3.126), we get

$$m\{x : |K_1 * u(x)| > \zeta\} \leq \left(\frac{|\mathfrak{S}|}{Q-\lambda}\right)^p \frac{\mu^{(Q-\lambda)p}}{\zeta^p}. \tag{3.128}$$

Then, similarly from Young's inequality, (2.11) and the assumptions, we obtain

$$
\begin{aligned}
\|K_2 * u\|_{L^\infty(\mathbb{G})} \leq \|K_2\|_{L^{p'}(\mathbb{G})} \|u\|_{L^p(\mathbb{G})} &= \left(\int_\mu^\infty r^{-\lambda p'} r^{Q-1} dr \int_{\mathfrak{S}} |y|^{-\lambda p'} d\sigma(y) \right)^{\frac{1}{p'}} \\
&= \left(\frac{|\mathfrak{S}|}{Q - \lambda p'} \right)^{\frac{1}{p'}} \left(\int_\mu^\infty r^{Q-\lambda p'-1} dr \right)^{\frac{1}{p'}} = \left(\frac{|\mathfrak{S}|}{Q - \lambda p'} \right)^{\frac{1}{p'}} (r^{Q-\lambda p'}|_\mu^\infty)^{\frac{1}{p'}} \\
&= \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}}, \quad (3.129)
\end{aligned}
$$

since from the assumptions, we get $\frac{Q-\lambda p'}{p'} = \frac{Q}{p'} - \lambda = Q(1 - \frac{1}{p} - \frac{\lambda}{Q}) = -\frac{Q}{q}$. Moreover, if $\left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{\frac{1}{p'}} \mu^{-\frac{Q}{q}} = \zeta$, then $\mu = \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{-\frac{q}{Qp'}} \zeta^{-\frac{\theta}{Q}}$, so we have $\|K_2 * u\|_{L^\infty(\mathbb{G})} \leq \zeta$. Hence, we have $m\{x : |K_2 * u| > \zeta\} = 0$. From these facts with (3.125), $\|u\|_{L^p(\mathbb{G})} = 1$ and the assumptions we get

$$
\begin{aligned}
m\{x : |K * u| > 2\zeta\} &\leq \left(\frac{|\mathfrak{S}|}{Q - \lambda} \right)^p \frac{\mu^{(Q-\lambda)p}}{\zeta^p} \\
&= \left(\frac{|\mathfrak{S}|}{Q - \lambda} \right)^p \left(\frac{|\mathfrak{S}|}{\lambda p' - Q} \right)^{-\frac{q(Q-\lambda)p}{Qp'}} \zeta^{\frac{-(Q-\lambda)pq}{Q} - p} \leq C \zeta^{\frac{-(Q-\lambda)pq}{Q} - p} = C \zeta^{(\frac{\lambda}{Q} - 1)pq - p} \\
&= C \zeta^{(\frac{1}{q} - \frac{1}{p})pq - p} = C \zeta^{p-q-p} = C \frac{\|u\|^q_{L^p(\mathbb{G})}}{\zeta^q}. \quad (3.130)
\end{aligned}
$$

For completeness, let us recall two well-known ingredients.

Definition 3.38 ([66]). Let $1 \leq p \leq \infty$, $1 \leq q < \infty$ and $V : L^p(\mathbb{G}) \to L^q(\mathbb{G})$ be a operator, then V is called an operator of *weak type* (p, q) if

$$
m\{x : |Vu| > \zeta\} \leq C \left(\frac{\|u\|_{L^p(\mathbb{G})}}{\zeta} \right)^q, \quad \zeta > 0, \tag{3.131}
$$

where C is a positive constant and independent by u.

Let us also recall the classical Marcinkiewicz interpolation theorem:

Theorem 3.39. *Let V be sublinear operator of weak type (p_k, q_k) with $1 \leq p_k \leq q_k < \infty$, $k = 0, 1$ and $q_0 < q_1$. Then V is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ with*

$$
\frac{1}{p} = \frac{1 - \gamma}{p_0} + \frac{\gamma}{p_1}, \quad \frac{1}{q} = \frac{1 - \gamma}{q_0} + \frac{\gamma}{q_1}, \tag{3.132}
$$

for any $0 < \gamma < 1$, namely,

$$
\|Vu\|_{L^q(\mathbb{G})} \leq C \|u\|_{L^p(\mathbb{G})}, \tag{3.133}
$$

for any $u \in L^p(\mathbb{G})$ and C is a positive constant.

By using assumptions $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{Q} - 1 < \frac{1}{p}$, we have $q > p$. According to Definition 3.38, $I_{\lambda, |\cdot|} u$ is of weak type (p, q), so by using the Marcinkiewicz interpolation theorem, we prove (3.122).

The proof of Theorem 3.37 is complete. ∎

Remark 3.40. *Under assumption of the Theorem 3.37 and $h \in L^{q'}(\mathbb{G})$, we have the following Hardy-Littlewood-Sobolev inequality*

$$\left|\int_{\mathbb{G}}\int_{\mathbb{G}} \frac{u(y)h(x)}{|y^{-1}x|^{\lambda}} dxdy\right| \leq C\|u\|_{L^p(\mathbb{G})}\|h\|_{L^{q'}(\mathbb{G})}, \tag{3.134}$$

where C is a positive constant independent of u and h.

3.8. **Stein-Weiss inequality.** In this section, we show the Stein-Weiss inequality on homogeneous Lie group. For showing this inequality we need some preliminary results as the integral version of Hardy inequalities on general homogeneous groups and Proposition 2.6 which is play key roles in our proof. Firstly, let us show the integral version of Hardy inequalities on general homogeneous groups.

Theorem 3.41 ([32]). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $1 < p \leq q < \infty$. Let $W(x)$ and $U(x)$, be positive functions on $\mathbb{G}$. Then we have the following properties:*

(1) The inequality

$$\left(\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(z)dz\right)^q W(x)dx\right)^{\frac{1}{q}} \leq C_1\left(\int_{\mathbb{G}} f^p(x)U(x)dx\right)^{\frac{1}{p}} \tag{3.135}$$

holds for all $f \geq 0$ a.e. on $\mathbb{G}$ if only if

$$A_1 := \sup_{R>0}\left(\int_{\mathbb{G}\backslash B(0,|x|)} W(x)dx\right)^{\frac{1}{q}}\left(\int_{B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty. \tag{3.136}$$

(2) The inequality

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,|x|)} f(z)dz\right)^q W(x)dx\right)^{\frac{1}{q}} \leq C_2\left(\int_{\mathbb{G}} f^p(x)U(x)dx\right)^{\frac{1}{p}}, \tag{3.137}$$

holds for all $f \geq 0$ if and only if

$$A_2 := \sup_{R>0}\left(\int_{B(0,|x|)} W(x)dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty. \tag{3.138}$$

(3) If $\{C_i\}_{i=1}^2$ are the smallest constants for which (3.135) and (3.137) hold, then

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_i, \quad i = 1, 2. \tag{3.139}$$

Now we formulate the Stein-Weiss inequality on $\mathbb{G}$.

Theorem 3.42. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Let $0 < \lambda < Q$, $1 < p < \infty$, $\alpha < \frac{Q}{p'}$, $\beta < \frac{Q}{q}$, $\alpha + \beta \geq 0$, $\frac{1}{q} = \frac{1}{p} + \frac{\alpha+\beta+\lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $1 < p \leq q < \infty$, we have*

$$\||x|^{-\beta} I_{\lambda,|\cdot|} u\|_{L^q(\mathbb{G})} \leq C\||x|^{\alpha} u\|_{L^p(\mathbb{G})}. \tag{3.140}$$

where C is positive constant and independent by u.

Proof. Let us define

$$\||x|^{-\beta}I_{\lambda,|\cdot|}u\|^q_{L^q(\mathbb{G})}=\int_{\mathbb{G}}\left(\int_{\mathbb{G}}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx=I_1+I_2+I_3, \tag{3.141}$$

where

$$I_1=\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx, \tag{3.142}$$

$$I_2=\int_{\mathbb{G}}\left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx, \tag{3.143}$$

and

$$I_3=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}dy}\right)^q dx. \tag{3.144}$$

From Proposition 2.6 we can suppose that our quasi-norm is actually a norm.

Step 1. Firstly, let us consider I_1. From Proposition 2.6 and the definition of the quasi-norm with $|y|\leq\frac{|x|}{2}$, we obtain

$$\begin{aligned}|x|=|x^{-1}|&=|x^{-1}yy^{-1}|\\ &\leq|x^{-1}y|+|y^{-1}|=|y^{-1}x|+|y|\\ &\leq|y^{-1}x|+\frac{|x|}{2}.\end{aligned}$$

For any $\lambda>0$, we get

$$2^{\lambda}|x|^{-\lambda}\geq|y^{-1}x|^{-\lambda}.$$

Therefore, we get

$$\begin{aligned}I_1=\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta}|y^{-1}x|^{\lambda}}dy\right)^q dx&\leq2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}\frac{u(y)}{|x|^{\beta+\lambda}}dy\right)^q dx\\ &=2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^q|x|^{-(\beta+\lambda)q}dx. \end{aligned}\tag{3.145}$$

Assume that $W(x)=|x|^{-(\beta+\lambda)q}$ and $U(y)=|y|^{\alpha p}$ and if condition (3.136) in Theorem 3.41 is satisfied, then by (3.135) we have

$$I_1\leq2^{\lambda}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^q|x|^{-(\beta+\lambda)q}dx\leq C_1\||x|^{\alpha}u\|^q_{L^p(\mathbb{G})}. \tag{3.146}$$

Let us check condition (3.136) with $W(x)=|x|^{-(\beta+\lambda)q}$ and $U(y)=|y|^{\alpha p}$. By the assumption we have $\alpha<\frac{Q}{p'}$, then

$$\frac{1}{q}=\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}-1<\frac{1}{p}+\frac{\frac{Q}{p'}+\beta+\lambda}{Q}-1=\frac{1}{p}+\frac{1}{p'}+\frac{\beta+\lambda}{Q}-1=\frac{\beta+\lambda}{Q},$$

that is, $Q-(\beta+\lambda)q<0$ and by the using polar decomposition (2.11):

$$\left(\int_{\mathbb{G}\setminus B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{G}\setminus B(0,|x|)} |x|^{-(\beta+\lambda)q}dx\right)^{\frac{1}{q}}$$
$$= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1}r^{-(\beta+\lambda)q}drd\sigma(y)\right)^{\frac{1}{q}} = \left(|\mathfrak{S}|\int_R^\infty r^{Q-1-(\beta+\lambda)q}dr\right)^{\frac{1}{q}} \le CR^{\frac{Q-(\beta+\lambda)q}{q}}. \tag{3.147}$$

From $\alpha<\frac{Q}{p'}$, we get

$$\alpha p(1-p')+Q>\alpha p(1-p')+\alpha p' = \alpha p+\alpha p'(1-p) = \alpha p-\alpha p = 0.$$

Finally, $\alpha p(1-p')+Q>0$. Then, let us consider

$$\left(\int_{B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int_{B(0,|x|)} |x|^{(1-p')\alpha p}dx\right)^{\frac{1}{p'}}$$
$$= \left(\int_0^R \int_{\mathfrak{S}} r^{(1-p')\alpha p}r^{Q-1}drd\sigma(y)\right)^{\frac{1}{p'}} \le C\left(|\mathfrak{S}|\int_0^R r^{(1-p')\alpha p+Q-1}dr\right)^{\frac{1}{p'}}$$
$$\le CR^{\frac{(1-p')\alpha p+Q}{p'}} = CR^{\frac{Q-\alpha p'}{p'}}. \tag{3.148}$$

Moreover, the assumptions imply

$$A_1 = \sup_{R>0}\left(\int_{\mathbb{G}\setminus B(0,|x|)} W(x)dx\right)^{\frac{1}{q}}\left(\int_{B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} \le CR^{\frac{Q-(\beta+\lambda)q}{q}+\frac{Q-\alpha p'}{p'}}$$
$$= CR^{Q\left(\frac{1}{q}-\frac{1}{p}-\frac{\alpha+\beta+\lambda}{Q}+1\right)} = C<\infty,$$

where $C=C(\alpha,\beta,p,\lambda)$ is a positive constant. From (3.135), we get

$$I_1 \le C\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^q |x|^{-(\beta+\lambda)q}dx \le C_1\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.149}$$

Step 2. Similarly with the previous case I_1, now we consider I_3. From $2|x|\le|y|$, we have

$$|y| = |y^{-1}| = |y^{-1}xx^{-1}| \le |y^{-1}x|+|x|$$
$$\le |y^{-1}x|+\frac{|y|}{2},$$

that is,

$$\frac{|y|}{2} \le |y^{-1}x|.$$

Then, if condition (3.138) with $W(x)=|x|^{-\beta q}$ and $U(y)=|y|^{(\alpha+\lambda)p}$ is satisfied, then we have

$$I_3 = \int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} \frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda}dy\right)^q dx \le C\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} \frac{u(y)}{|x|^\beta|y|^\lambda}dy\right)^q dx$$
$$= C\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)|y|^{-\lambda}dy\right)^q |x|^{-\beta q}dx \le C\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.150}$$

Now let us verify condition (3.138). Then, we get

$$\left(\int_{B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} = \left(\int_{B(0,|x|)} |x|^{-\beta q}dx\right)^{\frac{1}{q}}$$
$$= \left(\int_0^R \int_{\mathfrak{S}} r^{-\beta q} r^{Q-1} dr d\sigma(y)\right)^{\frac{1}{q}} \leq CR^{\frac{Q-\beta q}{q}}, \quad (3.151)$$

where $Q - \beta q > 0$, and

$$\left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} = \left(\int_{\mathbb{G}\backslash B(0,|x|)} |x|^{(\alpha+\lambda)(1-p')p}dx\right)^{\frac{1}{p'}}$$
$$= \left(\int_R^\infty \int_{\mathfrak{S}} r^{Q-1} r^{(\alpha+\lambda)(1-p')p} dr d\sigma(y)\right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'}}, \quad (3.152)$$

where from $\beta < \frac{Q}{q}$, we get $Q - p'(\alpha+\lambda) < 0$.

By using these facts we have

$$A_2 := \sup_{R>0} \left(\int_{B(0,|x|)} W(x)dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{G}\backslash B(0,|x|)} U^{1-p'}(x)dx\right)^{\frac{1}{p'}} \leq CR^{\frac{Q-p'(\alpha+\lambda)}{p'}+\frac{Q-\beta q}{q}}$$
$$= CR^{\frac{Q}{p'}-(\alpha+\beta+\lambda)+\frac{Q}{q}} = CR^{Q(\frac{1}{p'}-\frac{\alpha+\beta+\lambda}{Q}+\frac{1}{q})} = C < \infty, \quad (3.153)$$

where $C = C(\alpha, \beta, p, \lambda)$ is a positive constant. Then we establish

$$I_3 = \int_{\mathbb{G}} \left(\int_{\mathbb{G}\backslash B(0,2|x|)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy\right)^q dx \leq C \||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \quad (3.154)$$

Step 3. Let us estimate I_2 now.

Case 1: $p < q$. By $\frac{|x|}{2} < |y| < 2|x|$, we get

$$\frac{|y^{-1}x|}{2} \leq \frac{|x|+|y|}{2} = \frac{|x|}{2} + \frac{|y|}{2} < \frac{3}{2}|y|,$$

that is,

$$|y^{-1}x| < 3|y|.$$

For all $\alpha + \beta \geq 0$, we have

$$|y^{-1}x|^{\alpha+\beta} < 3^{\alpha+\beta}|y|^{\alpha+\beta} = 3^{\alpha+\beta}|y|^\alpha |y|^\beta \leq 3^{\alpha+\beta}2^{|\beta|}|x|^\beta |y|^\alpha.$$

Hence,

$$I_2 = \int_{\mathbb{G}} \left(\int_{B(0,2|x|)\backslash B\left(0,\frac{|x|}{2}\right)} \frac{u(y)}{|x|^\beta |y^{-1}x|^\lambda} dy\right)^q dx$$
$$\leq C \int_{\mathbb{G}} \left(\int_{B(0,2|x|)\backslash B\left(0,\frac{|x|}{2}\right)} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy\right)^q dx$$
$$\leq C \int_{\mathbb{G}} \left(\int_{\mathbb{G}} \frac{|y|^\alpha u(y)}{|y^{-1}x|^{\alpha+\beta+\lambda}} dy\right)^q dx = C\|I_{\lambda+\alpha+\beta,|\cdot|}\tilde{u}\|^q_{L^q(\mathbb{G})},$$

where $\tilde{u}(x) = |x|^\alpha u(x)$.

From the assumption $\frac{1}{q}-\frac{1}{p}=\frac{\lambda+\alpha+\beta}{Q}-1<0$, we get $Q>\lambda+\alpha+\beta$ and by using Theorem 3.37 with $p<q$, we obtain

$$I_2 \leq C\|I_{\lambda+\alpha+\beta,|\cdot|}\tilde{u}\|^q_{L^q(\mathbb{G})} \leq C\|\tilde{u}\|^q_{L^p(\mathbb{G})} = C\||x|^\alpha u\|^q_{L^p(\mathbb{G})}. \tag{3.155}$$

Case 2: $p=q$. Let us decompose I_2 as

$$I_2 = \sum_{k\in\mathbb{Z}} \int_{2^k\leq|x|\leq 2^{k+1}} \left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)} \frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda} dy \right)^p dx. \tag{3.156}$$

By $|x|\leq 2|y|\leq 4|x|$ and $2^k\leq|x|\leq 2^{k+1}$, we get $2^{k-1}\leq|y|\leq 2^{k+2}$ and $0\leq |y^{-1}x|\leq 3|x|\leq 3\cdot 2^{k+1}$.

By combining Young's inequality with $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$ (our case $p=q$, hence $r=1$), we get

$$\begin{aligned}
I_2 &= \sum_{k\in\mathbb{Z}} \int_{2^k\leq|x|\leq 2^{k+1}} \left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)} \frac{u(y)}{|x|^\beta|y^{-1}x|^\lambda} dy \right)^p dx \\
&= \sum_{k\in\mathbb{Z}} \int_{2^k\leq|x|\leq 2^{k+1}} \left(\int_{B(0,2|x|)\setminus B\left(0,\frac{|x|}{2}\right)} \frac{u(y)}{|y^{-1}x|^\lambda} dy \right)^p \frac{dx}{|x|^{\beta p}} \\
&\leq \sum_{k\in\mathbb{Z}} 2^{-\beta pk} \|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}} * |x|^{-\lambda}\|^p_{L^p(\mathbb{G})} \\
&\leq \sum_{k\in\mathbb{Z}} 2^{-\beta pk} \||x|^{-\lambda}\cdot\chi_{\{0\leq|y|\leq 3\cdot 2^{k+1}\}}\|^p_{L^1(\mathbb{G})} \|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} \\
&\leq C\sum_{k\in\mathbb{Z}} 2^{(Q-\lambda-\beta)kp} \|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} = C\sum_{k\in\mathbb{Z}} 2^{\alpha kp}\|u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} \\
&= C\sum_{k\in\mathbb{Z}} \|2^{\alpha(k-1)}u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} \leq C\sum_{k\in\mathbb{Z}} \||y|^\alpha u\cdot\chi_{\{2^{k-1}\leq|y|\leq 2^{k+2}\}}\|^p_{L^p(\mathbb{G})} \\
&= C\||x|^\alpha u\|^p_{L^p(\mathbb{G})}.
\end{aligned}$$

Theorem 3.42 is proved. ☐

Remark 3.43. *With assumptions Theorem 3.42 and $h\in L^{q'}(\mathbb{G})$, we have the following Stein-Weiss inequality*

$$\left| \int_{\mathbb{G}} \frac{u(y)h(x)}{|x|^\beta|y^{-1}x|^\lambda|y|^\alpha} dxdy \right| \leq C\|u\|_{L^p(\mathbb{G})}\|h\|_{L^{q'}(\mathbb{G})}, \tag{3.157}$$

where C is a positive constant independent of u and h.

Remark 3.44. *In inequality (3.140) with $\alpha=0$ we get the weighted Hardy-Littlewood-Sobolev inequality established in* [32, Theorem 4.1]. *Thus, by setting $\alpha=\beta=0$ we get Hardy-Littlewood-Sobolev inequality on the homogeneous Lie groups. In the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^N,+)$, we have $Q=N$ and $|\cdot|$ can be any homogeneous quasi-norm on $\mathbb{R}^N$, so with the usual Euclidean distance, i.e. $|\cdot|=\|\cdot\|_E$, Theorem 3.42 gives the classical result of Stein and Weiss.*

3.9. **Logarithmic Sobolev-Folland-Stein inequality.** In this section, we present the logarithmic Sobolev-Folland-Stein inequality on stratified groups. Let us recall the well-known Sobolev-Folland-Stein inequality.

Theorem 3.45. *Let $\mathbb{G}$ be a stratified Lie group and $\Omega \subset \mathbb{G}$ be an open set. Then there exists a constant $C_S = C_S(\mathbb{G}) > 0$ such that for all $u \in C_0^\infty(\Omega)$ we have*

$$\|u\|_{L^{p^*}(\Omega)} \leq C_S \left(\int_\Omega |\nabla_H u|^p dx \right)^{\frac{1}{p}}, \ 1 < p < Q, \tag{3.158}$$

where $p^* = \frac{Qp}{Q-p}$. Here ∇_H is the horizontal gradient and Q is the homogeneous dimension of $\mathbb{G}$.

Now let us state the logarithmic Sobolev-Folland-Stein inequality on stratified groups.

Theorem 3.46. *Suppose that $p^* = \frac{2Q}{Q-2}$ and $a > 0$. Then*

$$2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx + Q(1+\ln a)\|u\|^2_{L^2(\mathbb{G})} \leq QC_S^2 a \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})}, \tag{3.159}$$

where $u \in S_0^{1,2}(\mathbb{G})$.

Proof. By a direct calculation with $\varepsilon > 0$, we have

$$\begin{aligned} 2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &= \frac{1}{\varepsilon} \int_{\mathbb{G}} |u(x)|^2 \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx \\ &= \frac{\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \int_{\mathbb{G}} \frac{|u(x)|^2}{\|u\|^2_{L^2(\mathbb{G})}} \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx. \end{aligned} \tag{3.160}$$

From Jensen's inequality we obtain the upper estimate for the integral:

$$\begin{aligned} 2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &= \frac{2\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \int_{\mathbb{G}} \frac{|u(x)|^2}{\|u\|^2_{L^2(\mathbb{G})}} \ln \left(\frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} \right)^\varepsilon dx \\ &\leq \frac{2\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \left(\int_{\mathbb{G}} \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx \right)^{\varepsilon+1} \\ &= \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}}. \end{aligned} \tag{3.161}$$

From the inequality $\ln x \leq ax - \ln(a) - 1$ for all $a, x > 0$, and by choosing $2\varepsilon + 2 = \frac{2Q}{Q-2}$ as well as using the Sobolev-Folland-Stein inequality, we get

$$
\begin{aligned}
2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx &\leq \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \ln \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}} \\
&\leq \frac{2(\varepsilon+1)\|u\|^2_{L^2(\mathbb{G})}}{\varepsilon} \left(a \frac{\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})}}{\|u\|^2_{L^2(\mathbb{G})}} - (\ln(a)+1) \right) \\
&= \frac{2(\varepsilon+1)}{\varepsilon} \left(a\|u\|^2_{L^{2\varepsilon+2}(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&\leq \frac{2(\varepsilon+1)}{\varepsilon} \left(aC_S^2\|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&= \frac{\frac{2Q}{Q-2}}{\frac{2}{Q-2}} \left(aC_S^2\|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right) \\
&= Q \left(aC_S^2\|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})} - (\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \right).
\end{aligned} \tag{3.162}
$$

It yields that

$$
2\int_{\mathbb{G}} |u(x)|^2 \ln \frac{|u(x)|}{\|u\|_{L^2(\mathbb{G})}} dx + Q(\ln(a)+1)\|u\|^2_{L^2(\mathbb{G})} \leq C_S^2 Q a \|\nabla_{\mathbb{G}} u\|^2_{L^2(\mathbb{G})}. \tag{3.163}
$$

□

4. Reverse Inequalities

In this chapter, we show reverse integral Hardy inequality on metric measure space. We show the reverse integral Hardy inequality in two cases. In the first case we consider the case $q < 0$ and $p \in (0,1)$. In the second case, we consider the case $-\infty < q \leq p < 0$. For the both cases we also obtain conjugate reverse integral Hardy inequality. In the first case, as consequences we show the reverse integral Hardy inequality for the homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. Also, we show reverse Hardy-Littlewood-Sobolev and Stein-Weiss inequalities for the both cases. In addition, we obtain Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities on homogeneous groups with radial derivative.

Firstly, we need to give some preliminary results of this chapter. Let us recall briefly the reverse Hölder's inequality.

Theorem 4.1 ([35], Theorem 2.12, p. 27)**.** *Let $\mathbb{X}$ be metric measure space. Let $p \in (0,1)$, so that $p' = \frac{p}{p-1} < 0$. If non-negative functions satisfy $f \in L^p(\mathbb{X})$ and $0 < \int_{\mathbb{X}} g^{p'}(x)dx < +\infty$, we have*

$$\int_{\mathbb{X}} f(x)g(x)dx \geq \left(\int_{\mathbb{X}} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{X}} g^{p'}(x)dx\right)^{\frac{1}{p'}}. \tag{4.1}$$

Let us give the reverse integral Minkowski inequality (or a continuous version of reverse Minkowski inequality) with $q < 0$ on metric measure space.

Theorem 4.2. *Let $\mathbb{X}, \mathbb{Y}$ be metric measure spaces and let $F = F(x,y) \in \mathbb{X} \times \mathbb{Y}$ be a non-negative measurable function. Then we have*

$$\left[\int_{\mathbb{X}} \left(\int_{\mathbb{Y}} F(x,y)dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}} \left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}} dy, \; q < 0. \tag{4.2}$$

Proof. Let us consider the following function:

$$A(x) := \int_{\mathbb{Y}} F(x,y)dy, \tag{4.3}$$

so we have

$$A^q(x) = \left(\int_{\mathbb{Y}} F(x,y)dy\right)^q. \tag{4.4}$$

By integrating over $\mathbb{X}$ both sides and by using reverse Hölder's inequality (Theorem 4.1), we obtain

$$\begin{aligned}
\int_{\mathbb{X}} A^q(x)dx &= \int_{\mathbb{X}} A^{q-1}(x)A(x)dx \\
&= \int_{\mathbb{X}} A^{q-1}(x)\int_{\mathbb{Y}} F(x,y)dydx \\
&= \int_{\mathbb{Y}}\int_{\mathbb{X}} A^{q-1}(x)F(x,y)dxdy \\
&\overset{(4.1)}{\geq} \int_{\mathbb{Y}}\left(\int_{\mathbb{X}} A^{q-1\frac{q}{q-1}}(x)dx\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy \\
&= \left(\int_{\mathbb{X}} A^q(x)dx\right)^{\frac{q-1}{q}}\int_{\mathbb{Y}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy.
\end{aligned} \tag{4.5}$$

From this, we get

$$\left[\int_{\mathbb{X}}\left(\int_{\mathbb{Y}} F(x,y)dy\right)^q dx\right]^{\frac{1}{q}} \geq \int_{\mathbb{Y}}\left(\int_{\mathbb{X}} F^q(x,y)dx\right)^{\frac{1}{q}}dy, \tag{4.6}$$

proving (4.2). □

Remark 4.3. *In our sense, the negative exponent $q<0$ of 0, we understand in the following form:*

$$0^q = (+\infty)^{-q} = +\infty, \quad \textit{and} \quad 0^{-q} = (+\infty)^q = 0. \tag{4.7}$$

We denote by $B(a,r)$ the ball in $\mathbb{X}$ with centre a and radius r, i.e

$$B(a,r) := \{x \in \mathbb{X} : d(x,a) < r\},$$

where d is the metric on $\mathbb{X}$. Once and for all we will fix some point $a \in \mathbb{X}$, and we will write

$$|x|_a := d(a,x). \tag{4.8}$$

4.1. **Reverse integral Hardy inequality with $q<0$ and $p\in(0,1)$ on the metric measure space.** Now we prove the reverse integral Hardy inequality on a metric measure space.

Theorem 4.4 (Reverse integral Hardy inequality)**.** *Suppose that $p\in(0,1)$ and $q<0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a\in\mathbb{X}$. Assume that $u,v>0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.9}$$

holds for some $C(p,q)>0$ and for all non-negative real-valued measurable functions f, if and only if

$$0 < D_1 := \inf_{x\neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.10}$$

Moreover, the biggest constant $C(p,q)$ in (4.9) has the following relation to D_1:

$$D_1 \geq C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}} \left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1. \tag{4.11}$$

Proof. Let us divide proof of this theorem in several steps.

Step 1. Let us denote $g(x) := f(x)v^{\frac{1}{p}}(x)$. Let $\frac{1}{p}+\frac{1}{p'}=1$, $\alpha \in \left(0, -\frac{1}{p'}\right)$ and $z(x) = v^{-\frac{1}{p}}(x)$. Let us denote,

$$V(x) := \int_{B(a,|x|_a)} v^{-\frac{p'}{p}}(y)dy = \int_{B(a,|x|_a)} z^{p'}(y)dy, \tag{4.12}$$

$$H_1(s) := \int_{\sum_s} \lambda(s,\sigma)g(s,\sigma)z(s,\sigma)d\sigma, \tag{4.13}$$

$$H_2(s) := \int_{\sum_s} \lambda(s,\sigma)z^{p'}(s,\sigma)V^{\alpha p'}(s,\sigma)d\sigma, \tag{4.14}$$

$$H_3(s) := \int_{\sum_s} \lambda(s,\sigma)g^{p}(s,\sigma)V^{-\alpha p}(s,\sigma)d\sigma, \tag{4.15}$$

$$U(r) := \int_{\sum_r} \lambda(r,\omega)u(r,\omega)d\omega. \tag{4.16}$$

By using reverse Hölder's inequality (Theorem 4.1) with polar decomposition (2.31), we compute

$$\begin{aligned}
A :&= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx = \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^q u(x)dx \\
&= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^p \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\
&= \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g(y)V^{-\alpha}(y)V^{\alpha}(y)z(y)dy\right)^p \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\
&\geq \int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} g^p(y)V^{-\alpha p}(y)dy\right) \left(\int_{B(a,|x|_a)} z^{p'}(y)V^{\alpha p'}(y)dy\right)^{\frac{p}{p'}} \\
&\times \left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx \\
&= \int_0^{\infty} U(r)\left(\int_0^r H_1(s)ds\right)^{q-p} \left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} \left(\int_0^r H_3(s)ds\right) dr.
\end{aligned} \tag{4.17}$$

Let us denote by $\tilde{H}_2(s) := \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) d\sigma$. Then we obtain

$$
\begin{aligned}
\left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} &= \left(\int_0^r \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) V^{\alpha p'}(s,\sigma) ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) \left(\int_0^s \int_{\sum_\rho} \lambda(\rho,\sigma_1) z^{p'}(\rho,\sigma_1) d\rho d\sigma_1\right)^{\alpha p'} ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_0^r \tilde{H}_2(s) \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{\alpha p'} ds\right)^{\frac{p}{p'}} \qquad (4.18) \\
&= \left(\int_0^r \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{\alpha p'} d_s \left(\int_0^s \tilde{H}_2(\rho)d\rho\right)\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\left(\int_0^s \tilde{H}_2(\rho)d\rho\right)^{1+\alpha p'} \Big|_0^r\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^r \tilde{H}_2(\rho)d\rho\right)^{\frac{p(1+\alpha p')}{p'}} \\
&= \frac{V_1^{\frac{p(1+\alpha p')}{p'}}(r)}{(1+\alpha p')^{\frac{p}{p'}}},
\end{aligned}
$$

where $V_1(r) = \int_0^r \tilde{H}_2(\rho)d\rho$. By combining this fact and reverse Hölder's inequality with $\frac{p}{q} + \frac{q-p}{q} = 1$, we get

$$
\begin{aligned}
A &\geq \int_0^\infty \left(\int_0^r H_3(s)ds\right) U(r) \left(\int_0^r H_1(s)ds\right)^{q-p} \left(\int_0^r H_2(s)ds\right)^{\frac{p}{p'}} dr \\
&\overset{(4.18)}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty \left(\int_0^r H_3(s)ds\right) U(r) \left(\int_0^r H_1(s)ds\right)^{q-p} V_1^{\frac{p(1+\alpha p')}{p'}}(r)dr \\
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty U^{\frac{p}{q}}(r) \left(\int_0^r H_3(s)ds\right) V_1^{\frac{p(1+\alpha p')}{p'}}(r) \left(\int_0^r H_1(s)ds\right)^{q-p} U^{\frac{q-p}{q}} dr \\
&\geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty \left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} U(r) V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}} \\
&\times \left(\int_0^\infty \left(\int_0^r H_1(s)ds\right)^q U(r)dr\right)^{\frac{q-p}{q}}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}\\
&\times\left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} g(y)z(y)dy\right)^q u(x)dx\right)^{\frac{q-p}{q}}\\
=&\frac{A^{\frac{q-p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}.
\end{aligned}
$$

Therefore,

$$
A^{\frac{p}{q}}\geq\frac{1}{(1+\alpha p')^{\frac{p}{p'}}}\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}.
$$

By using the reverse Minkowski inequality (continuous version of reverse Minkowski inequality) with exponent $\frac{q}{p}<0$, then we obtain

$$
\begin{aligned}
&\left(\int_0^\infty U(r)\left(\int_0^r H_3(s)ds\right)^{\frac{q}{p}} V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}\\
&=\left(\int_0^\infty\left(\int_0^r U^{\frac{p}{q}}(r)H_3(s)V^{\frac{(1+\alpha p')p}{p'}}(r)ds\right)^{\frac{q}{p}}dr\right)^{\frac{p}{q}}\\
&=\left(\int_0^\infty\left(\int_0^\infty U^{\frac{p}{q}}(r)H_3(s)V_1^{\frac{(1+\alpha p')p}{p'}}(r)\chi_{\{s<r\}}ds\right)^{\frac{q}{p}}dr\right)^{\frac{p}{q}}\\
&\overset{(4.2)}{\geq}\int_0^\infty H_3(s)\left(\int_s^\infty U(r)V_1^{\frac{q(1+\alpha p')}{p'}}(r)dr\right)^{\frac{p}{q}}ds\\
&=\int_{\mathbb{X}} g^p(y)V^{-\alpha p}(y)\left(\int_{\mathbb{X}\backslash B(a,|y|_a)} u(x)V^{\frac{q(1+\alpha p')}{p'}}(x)dx\right)^{\frac{p}{q}}dy\\
&\geq D^p(\alpha)\int_{\mathbb{X}} g^p(y)dy,
\end{aligned}
$$

where $D(\alpha):=\inf\limits_{x\neq a} D(x,\alpha)=\inf\limits_{x\neq a} V^{-\alpha}(x)\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} u(y)V^{\frac{q(1+\alpha p')}{p'}}(y)dy\right)^{\frac{1}{q}}$ and χ is the cut-off function. Then we have

$$
\begin{aligned}
A^{\frac{p}{q}}&=\left(\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right)^{\frac{p}{q}}\geq\frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}}\int_{\mathbb{X}} g^p(y)dy\\
&=\frac{D^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}}\int_{\mathbb{X}} f^p(y)v(y)dy.
\end{aligned}
$$

Step 2. Let us define D_1 in the following form:

$$
0<D_1=\inf_{x\neq a}\left[\left(\int_{\mathbb{X}\backslash B(a,|x|_a)} u(x)dx\right)^{\frac{1}{q}}\left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.19}
$$

Let us note a relation between V and V_1,

$$
\begin{aligned}
V(x) = \int_{B(a,|x|_a)} v^{-\frac{p'}{p}} dx &= \int_{B(a,|x|_a)} z^{p'} dx \\
&= \int_0^{|x|_a} \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega) dr d\omega \\
&= \int_0^{|x|_a} \tilde{H}_2(r) dr \\
&=: V_1(|x|_a),
\end{aligned} \tag{4.20}
$$

where, as before, $\tilde{H}_2(r) = \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega)d\omega$. Then let us calculate the following integral:

$$
\begin{aligned}
I &= \int_{\mathbb{X}\setminus B(a,|x|_a)} u(y) V^{\frac{q(1+\alpha p')}{p'}}(y) dy = \int_{|x|_a}^{\infty} \int_{\sum_r} \lambda(r,\omega) u(r,\omega) V_1^{\frac{q(1+\alpha p')}{p'}}(r) dr d\omega \\
&= \int_{|x|_a}^{\infty} U(r) V_1^{\frac{q(1+\alpha p')}{p'}}(r) dr = \int_{|x|_a}^{\infty} V_1^{\frac{q(1+\alpha p')}{p'}}(r) d_r \left(- \int_r^{\infty} U(s) ds \right) \\
&= -V_1^{\frac{q(1+\alpha p')}{p'}}(r) \int_r^{\infty} U(s) ds \Big|_{|x|_a}^{\infty} \\
&+ \frac{q(1+\alpha p')}{p'} \int_{|x|_a}^{\infty} \left(\int_r^{\infty} U(s) ds \right) V_1^{\frac{q(1+\alpha p')}{p'} - 1}(r) dV_1(r) \\
&\overset{\frac{q}{p'}>0}{=} V_1^{\frac{q(1+\alpha p')}{p'}}(|x|_a) \int_{|x|_a}^{\infty} U(s) ds \\
&+ \frac{q(1+\alpha p')}{p'} \int_{|x|_a}^{\infty} \left(\int_r^{\infty} U(s) ds \right) V_1^{\frac{q(1+\alpha p')}{p'} - 1}(r) dV_1(r) \\
&= V_1^{\frac{q}{p'}}(|x|_a) \left(\int_{|x|_a}^{\infty} U(s) ds \right) V_1^{\alpha q}(|x|_a) \\
&+ \frac{q(1+\alpha p')}{p'} \int_{|x|_a}^{\infty} \left(\int_r^{\infty} U(s) ds \right) V_1^{\frac{q}{p'}}(r) V_1^{\alpha q - 1}(r) dV_1(r) \\
&\le D_1^q V_1^{\alpha q}(|x|_a) + \frac{q(1+\alpha p') D_1^q}{p'} \int_{|x|_a}^{\infty} V_1^{\alpha q - 1}(r) dV_1(r) \\
&= D_1^q V_1^{\alpha q}(|x|_a) + \frac{(1+\alpha p') D_1^q}{\alpha p'} V_1^{\alpha q}(r) \Big|_{|x|_a}^{\infty} \\
&= D_1^q V_1^{\alpha q}(|x|_a) + \lim_{r\to\infty} \frac{(1+\alpha p') D_1^q}{\alpha p'} V_1^{\alpha q}(r) - \frac{(1+\alpha p') D_1^q}{\alpha p'} V_1^{\alpha q}(|x|_a) \\
&\overset{\frac{(1+\alpha p')D_1^q}{\alpha p'}<0}{\le} D_1^q V_1^{\alpha q}(|x|_a) - \frac{(1+\alpha p') D_1^q}{\alpha p'} V_1^{\alpha q}(|x|_a) \\
&\overset{(4.20)}{=} -\frac{1}{\alpha p'} D_1^q V^{\alpha q}(x).
\end{aligned} \tag{4.21}
$$

Then we get $I = D^q(x,\alpha)V^{\alpha q}(x) \leq -\frac{1}{\alpha p'} D_1^q V^{\alpha q}(x)$. Hence,

$$D(x,\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1,$$

it means

$$D(\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_1.$$

Finally, we obtain

$$A^{\frac{1}{q}} \geq \frac{D_1(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} \left(\int_{\mathbb{X}} f^p(y)v(y)dy \right)^{\frac{1}{p}}.$$

Let us consider the function $k(\alpha) := \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} = (-\alpha p')^{-\frac{1}{q}}(1+\alpha p')^{-\frac{1}{p'}}$, where $\alpha \in \left(0, -\frac{1}{p'}\right)$. Firstly, let us find extremum of this function. After some calculation we have

$$\begin{aligned} \frac{dk(\alpha)}{d\alpha} &= -\frac{1}{q}(-p')(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}} \\ &+ \left(-\frac{1}{p'}\right) p'(1+\alpha p')^{-\frac{1}{p'}-1}(-\alpha p')^{-\frac{1}{q}} \\ &= p'(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1} \left(\frac{(1+\alpha p')}{q} + \alpha \right) \\ &= \frac{p'}{q}(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1} \left(\alpha(p'+q)+1\right) \\ &= 0, \end{aligned} \tag{4.22}$$

it implies that its solution is

$$\alpha_1 = -\frac{1}{p'+q} \in \left(0, -\frac{1}{p'}\right).$$

After taking the second derivative of $k(\alpha)$ at the point α_1 and by denoting $k_1(\alpha) = (-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}$, we get

$$
\begin{aligned}
\frac{d^2k(\alpha)}{d\alpha^2}\Big|_{\alpha=\alpha_1} &= \left(\frac{p'}{q}(-\alpha p')^{-\frac{1}{q}-1}(1+\alpha p')^{-\frac{1}{p'}-1}\left(\alpha(p'+q)+1\right)\right)'\Big|_{\alpha=\alpha_1} \\
&= \frac{p'}{q}\left(k_1(\alpha)\left(\alpha(p'+q)+1\right)\right)'\Big|_{\alpha=\alpha_1} \\
&= \frac{p'}{q}\left(\frac{dk_1(\alpha)}{d\alpha}\left(\alpha(p'+q)+1\right)+(p'+q)k_1(\alpha)\right)\Big|_{\alpha=\alpha_1} \\
&= \frac{p'}{q}\left(\frac{dk_1(\alpha)}{d\alpha}\Big|_{\alpha=\alpha_1}\underbrace{\left(\alpha_1(p'+q)+1\right)}_{=0}+(p'+q)k_1(\alpha_1)\right) \\
&= \frac{p'(p'+q)}{q}k_1(\alpha_1) = \frac{p'(p'+q)}{q}(-\alpha_1 p')^{-\frac{1}{q}-1}(1+\alpha_1 p')^{-\frac{1}{p'}-1} \\
&= \underbrace{\frac{p'(p'+q)}{q}}_{<0}\underbrace{\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}-1}}_{>0}\underbrace{\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}-1}}_{>0} < 0.
\end{aligned} \tag{4.23}
$$

It means, function $k(\alpha)$ has supremum at the point $\alpha = \alpha_1$. Then, the biggest constant has the following relationship $C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_1$.

Step 3. Let us give a necessity condition of inequality (4.9). By using (4.9) and $f(x) = v^{-\frac{p'}{p}}(x)\chi_{\{(0,t)\}}(|x|_a)$, we compute

$$
\begin{aligned}
C(p,q) &\leq \left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{\mathbb{X}} f^p(y)v(y)dx\right]^{-\frac{1}{p}} \\
&= \left[\int_{\mathbb{X}}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{-\frac{1}{p}} \\
&\overset{q<0}{\leq} \left[\int_{|x|_a\geq t}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{-\frac{1}{p}} \\
&= \left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}}\left[\int_{|y|_a\leq t} v^{-p'}(y)v(y)dx\right]^{\frac{1}{p'}},
\end{aligned} \tag{4.24}
$$

which gives $D_1 \geq C(p,q)$. ◻

Let us give conjugate reverse integral Hardy inequality.

Theorem 4.5 (Conjugate reverse integral Hardy inequality)**.** *Suppose that $p \in (0,1)$ and $q < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at a. Assume that $u, v > 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$
\left[\int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.25}
$$

holds for some $C(p,q)>0$ and for all non-negative real-valued measurable functions f, if only if

$$0 < D_2 := \inf_{x\neq a}\left[\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right]. \tag{4.26}$$

Moreover, the biggest constant $C(p,q)$ in (4.25) has the following relation to D_2:

$$D_2 \geq C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_2. \tag{4.27}$$

Proof. Proof of this theorem is similar to the previous case. Let us divide proof of this theorem in several steps.

Step 1. Let us denote $g(x) := f(x)v^{\frac{1}{p}}(x)$. Let $\frac{1}{p}+\frac{1}{p'}=1$, $\alpha\in\left(0,-\frac{1}{p'}\right)$ and $z(x)=v^{-\frac{1}{p}}(x)$. Let us denote,

$$G(x) := \int_{\mathbb{X}\setminus B(a,|x|_a)} v^{-\frac{p'}{p}}(y)dy = \int_{\mathbb{X}\setminus B(a,|x|_a)} z^{p'}(y)dy.$$

By using reverse Hölder's inequality (Theorem 4.1), we get

$$\begin{aligned}
B :&= \int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx = \int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy\right)^q u(x)dx\\
&= \int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy\right)^p\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx\\
&= \int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)G^{-\alpha}(y)G^{\alpha}(y)z(y)dy\right)^p\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx\\
&\geq \int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g^p(y)G^{-\alpha p}(y)dy\right)\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} z^{p'}(y)G^{\alpha p'}(y)dy\right)^{\frac{p}{p'}}\\
&\times\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy\right)^{q-p} u(x)dx\\
&= \int_0^\infty U(r)\left(\int_r^\infty H_1(s)ds\right)^{q-p}\left(\int_r^\infty H_2(s)ds\right)^{\frac{p}{p'}}\left(\int_r^\infty H_3(s)ds\right)dr,
\end{aligned} \tag{4.28}$$

where $U(r), H_i(s)$, $i=1,2,3$, are defined in (4.13)-(4.16). Let us denote by $\tilde{H}_2(s) := \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) d\sigma$. Then we have

$$\begin{aligned}
\left(\int_r^\infty H_2(s)ds\right)^{\frac{p}{p'}} &= \left(\int_r^\infty \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) V^{\alpha p'}(s,\sigma) ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_r^\infty \int_{\sum_s} \lambda(s,\sigma) z^{p'}(s,\sigma) \left(\int_r^\infty \int_{\sum_\rho} \lambda(\rho,\sigma_1) z^{p'}(\rho,\sigma_1) d\rho d\sigma_1\right)^{\alpha p'} ds d\sigma\right)^{\frac{p}{p'}} \\
&= \left(\int_r^\infty \tilde{H}_2(s) \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} ds\right)^{\frac{p}{p'}} \qquad (4.29) \\
&= \left(\int_r^\infty \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} d_s\left(-\int_s^\infty \tilde{H}_2(\rho) d\rho\right)\right)^{\frac{p}{p'}} \\
&= \left(-\int_r^\infty \left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{\alpha p'} d_s\left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(-\left(\int_s^\infty \tilde{H}_2(\rho) d\rho\right)^{1+\alpha p'} \Big|_r^\infty\right)^{\frac{p}{p'}} \\
&\overset{1+\alpha p'>0}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_r^\infty \tilde{H}_2(\rho) d\rho\right)^{\frac{p(1+\alpha p')}{p'}} \\
&= \frac{G_1^{\frac{p(1+\alpha p')}{p'}}(r)}{(1+\alpha p')^{\frac{p}{p'}}},
\end{aligned}$$

where $G_1(r) = \int_0^r \tilde{H}_2(\rho) d\rho$. By combining this fact and reverse Hölder's inequality with $\frac{p}{q} + \frac{q-p}{q} = 1$, we get

$$\begin{aligned}
B &\geq \int_0^\infty \left(\int_r^\infty H_3(s)ds\right) U(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} \left(\int_r^\infty H_2(s)ds\right)^{\frac{p}{p'}} dr \\
&\overset{(4.29)}{=} \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty \left(\int_r^\infty H_3(s)ds\right) U(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} G_1^{\frac{p(1+\alpha p')}{p'}}(r) dr \\
&= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \int_0^\infty U^{\frac{p}{q}}(r) \left(\int_r^\infty H_3(s)ds\right) G_1^{\frac{p(1+\alpha p')}{p'}}(r) \left(\int_r^\infty H_1(s)ds\right)^{q-p} U^{\frac{q-p}{q}} dr \\
&\geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty \left(\int_r^\infty H_3(s)ds\right)^{\frac{q}{p}} U(r) G_1^{\frac{q(1+\alpha p')}{p'}}(r) dr\right)^{\frac{p}{q}} \\
&\times \left(\int_0^\infty \left(\int_r^\infty H_1(s)ds\right)^{q} U(r) dr\right)^{\frac{q-p}{q}}
\end{aligned}$$

$$= \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}$$

$$\times \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} g(y)z(y)dy \right)^q u(x)dx \right)^{\frac{q-p}{q}}$$

$$= \frac{B^{\frac{q-p}{q}}}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}.$$

Therefore,

$$B^{\frac{p}{q}} \geq \frac{1}{(1+\alpha p')^{\frac{p}{p'}}} \left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}. \tag{4.30}$$

From reverse Minkowski inequality with exponent $\frac{q}{p} < 0$, we obtain

$$\left(\int_0^\infty U(r) \left(\int_r^\infty H_3(s)ds \right)^{\frac{q}{p}} G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}}$$

$$= \left(\int_0^\infty \left(\int_r^\infty U^{\frac{p}{q}}(r)H_3(s)G^{\frac{(1+\alpha p')p}{p'}}(r)ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}}$$

$$= \left(\int_0^\infty \left(\int_0^\infty U^{\frac{p}{q}}(r)H_3(s)G_1^{\frac{(1+\alpha p')p}{p'}}(r)\chi_{\{r<s\}}ds \right)^{\frac{q}{p}} dr \right)^{\frac{p}{q}}$$

$$\overset{(4.2)}{\geq} \int_0^\infty H_3(s) \left(\int_0^s U(r)G_1^{\frac{q(1+\alpha p')}{p'}}(r)dr \right)^{\frac{p}{q}} ds$$

$$= \int_{\mathbb{X}} g^p(y)G^{-\alpha p}(y) \left(\int_{\mathbb{X}\setminus B(a,|y|_a)} u(x)G^{\frac{q(1+\alpha p')}{p'}}(x)dx \right)^{\frac{p}{q}} dy$$

$$\geq \widetilde{D}^p(\alpha) \int_{\mathbb{X}} g^p(y)dy,$$

where $\widetilde{D}(\alpha) := \inf_{x\neq a} \widetilde{D}(x,\alpha) = \inf_{x\neq a} G^{-\alpha}(x) \left(\int_{B(a,|x|_a)} u(y)G^{\frac{q(1+\alpha p')}{p'}}(y)dy \right)^{\frac{1}{q}}$ and χ is the cut-off function. Then we obtain

$$B^{\frac{p}{q}} = \left(\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy \right)^q u(x)dx \right)^{\frac{p}{q}} \geq \frac{\widetilde{D}^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} g^p(y)dy$$

$$= \frac{\widetilde{D}^p(\alpha)}{(1+\alpha p')^{\frac{p}{p'}}} \int_{\mathbb{X}} f^p(y)v(y)dy.$$

Step 2. Let us define D_2 in the following form:

$$0 < D_2 = \inf_{x\neq a} \left[\left(\int_{B(a,|x|_a)} u(x)dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy \right)^{\frac{1}{p'}} \right]. \tag{4.31}$$

Let us note a relation between G and G_1,

$$\begin{aligned} G(x) = \int_{\mathbb{X}\setminus B(a,|x|_a)} v^{-\frac{p'}{p}} dx &= \int_{\mathbb{X}\setminus B(a,|x|_a)} z^{p'} dx \\ &= \int_{|x|_a}^{\infty} \int_{\sum_r} z^{p'}(r,\omega)\lambda(r,\omega) dr d\omega \\ &= \int_{|x|_a}^{\infty} \tilde{H}_2(r) dr \\ &=: G_1(|x|_a). \end{aligned} \tag{4.32}$$

For $|x|_a \leq |y|_a$, we have

$$G_1(|x|_a) = \int_{|x|_a}^{\infty} \tilde{H}_2(r) dr \geq \int_{|y|_a}^{\infty} \tilde{H}_2(r) dr = G_1(|y|_a),$$

it means $G(x) \geq G(y)$. By $\frac{q(1+\alpha p')}{p'} > 0$, we get

$$\int_{B(a,|x|_a)} u(y) G^{\frac{q(1+\alpha p')}{p'}}(x) dy \geq \int_{B(a,|x|_a)} u(y) G^{\frac{q(1+\alpha p')}{p'}}(y) dy,$$

and by using $q < 0$, we have

$$\begin{aligned} \widetilde{D}(x,\alpha) &= G^{-\alpha}(x) \left(\int_{B(a,|x|_a)} u(y) G^{\frac{q(1+\alpha p')}{p'}}(y) dy \right)^{\frac{1}{q}} \\ &\geq G^{-\alpha}(x) \left(\int_{B(a,|x|_a)} u(y) G^{\frac{q(1+\alpha p')}{p'}}(x) dy \right)^{\frac{1}{q}} \\ &= G^{-\alpha}(x) G^{\frac{(1+\alpha p')}{p'}}(x) \left(\int_{B(a,|x|_a)} u(y) dy \right)^{\frac{1}{q}} \\ &= G^{\frac{1}{p'}}(x) \left(\int_{B(a,|x|_a)} u(y) dy \right)^{\frac{1}{q}} \\ &\overset{1>(-\alpha p')^{-\frac{1}{q}}}{\geq} (-\alpha p')^{-\frac{1}{q}} G^{\frac{1}{p'}}(x) \left(\int_{B(a,|x|_a)} u(y) dy \right)^{\frac{1}{q}}. \end{aligned} \tag{4.33}$$

Consequently,

$$\widetilde{D}(x,\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_2,$$

it means

$$\widetilde{D}(\alpha) \geq (-\alpha p')^{-\frac{1}{q}} D_2.$$

Finally, we obtain

$$B^{\frac{1}{q}} \geq \frac{D_1 (-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} \left(\int_{\mathbb{X}} f^p(y) v(y) dy \right)^{\frac{1}{p}}.$$

Then, as in the previous case we have

$$\sup_{\alpha \in \left(0, -\frac{1}{p'}\right)} \frac{(-\alpha p')^{-\frac{1}{q}}}{(1+\alpha p')^{\frac{1}{p'}}} = \left(\frac{p'}{p'+q} \right)^{-\frac{1}{q}} \left(\frac{q}{p'+q} \right)^{-\frac{1}{p'}}.$$

Therefore, we have that the biggest constant satisfies

$$C(p,q) \geq \left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}} \left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}} D_2.$$

Step 3. Let us give a necessity condition of inequality (4.25). By using (4.25) and $f(x) = v^{-\frac{p'}{p}}(x)\chi_{(t,\infty)}(|x|_a)$, where χ is cut-off function, we compute

$$\begin{aligned}
C(p,q) &\leq \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{\mathbb{X}} f^p(y)v(y)dx\right)^{-\frac{1}{p}} \\
&= \left[\int_{\mathbb{X}} \left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{|x|_a\geq t} v^{-p'}(y)v(y)dx\right)^{-\frac{1}{p}} \\
&\overset{q<0}{\leq} \left[\int_{|x|_a\leq t} \left(\int_{|x|_a\geq t} v^{-\frac{p'}{p}}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \left(\int_{|x|_a\geq t} v^{-p'}(y)v(y)dx\right)^{-\frac{1}{p}} \\
&= \left[\int_{|x|_a\geq t} v^{1-p'}(y)dx\right]^{\frac{1}{p'}} \left[\int_{|x|_a\leq t} u(y)dy\right]^{\frac{1}{q}},
\end{aligned} \tag{4.34}$$

which gives $D_2 \geq C(p,q)$. $\square$

4.2. **Reverse integral Hardy inequality with $-\infty < q \leq p < 0$ on the metric measure space.** Main results of this section we show the reverse integral Hardy inequality and its conjugate in the case $-\infty < q \leq p < 0$.

Theorem 4.6. *Assume that $p, q < 0$ such that $q \leq p < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v \geq 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$\left[\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C_1(p,q) \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{4.35}$$

holds for all non-negative real-valued measurable functions f, if

$$0 < D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|_a) = \inf_{x\neq a} \left[\left(\int_{B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \tag{4.36}$$

and $\mathcal{D}_1(|x|_a)$ is a non-decreasing. Moreover, biggest constant $C_1(p,q)$ satisfies

$$D_1 \geq C_1(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1, \tag{4.37}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Similarly to the previous case, let us divide proof of this theorem by steps.

Step 1. Firstly, by denoting

$$F_n(s) := \int_{\sum_\sigma} \lambda(s,\sigma) f^p(s,\sigma) v(s,\sigma) d\sigma, \tag{4.38}$$

$$V_n(s) := \int_{\sum_\sigma} \lambda(s,\sigma) v^{1-p'}(s,\sigma) d\sigma, \tag{4.39}$$

$$h(t) := \left(\int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{1-p'}(s,\sigma) ds d\sigma \right)^{\frac{1}{pp'}}, \tag{4.40}$$

$$H_1(t) := \int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) d\sigma ds, \tag{4.41}$$

$$U_1(t) := \int_{\sum_\sigma} \lambda(s,\sigma) u(s,\sigma) d\sigma. \tag{4.42}$$

By using the reverse Hölder's inequality with the polar decomposition, we compute

$$\begin{aligned}
\int_{B(a,|x|_a)} f(y)dy &= \int_{B(a,|x|_a)} [f(y) v^{\frac{1}{p}}(y) h(y)][v^{\frac{1}{p}}(y) h(y)]^{-1} dy \\
&\geq \left(\int_{B(a,|x|_a)} (f(y) v^{\frac{1}{p}}(y) h(y))^p dy \right)^{\frac{1}{p}} \left(\int_{B(a,|x|_a)} (v^{\frac{1}{p}}(y) h(y))^{-p'} dy \right)^{\frac{1}{p'}} \\
&= \left(\int_0^r \int_{\sum_\sigma} h^p(s) \lambda(s,\sigma) f^p(s,\sigma) v(s,\sigma) d\sigma ds \right)^{\frac{1}{p}} \\
&\times \left(\int_0^r \int_{\sum_\sigma} v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) \lambda(s,\sigma) d\sigma ds \right)^{\frac{1}{p'}} \\
&= \left(\int_0^r h^p(s) F_n(s) ds \right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(r).
\end{aligned} \tag{4.43}$$

Let us calculate the $H_1(t)$, then we obtain

$$\begin{aligned}
H_1(t) &= \int_0^t \int_{\sum_\sigma} \lambda(s,\sigma) v^{-\frac{p'}{p}}(s,\sigma) h^{-p'}(s) d\sigma ds \overset{(4.39)}{=} \int_0^t h^{-p'}(s) V_n(s) ds \\
&\overset{(4.40)}{=} \int_0^t \left(\int_0^s \int_{\sum_z} \lambda(z,\omega) v^{1-p'}(z,\omega) dz d\omega \right)^{-\frac{1}{p}} V_n(s) ds \\
&\overset{(4.39)}{=} \int_0^t \left(\int_0^s V_n(z) dz \right)^{-\frac{1}{p}} V_n(s) ds \\
&= \int_0^t \left(\int_0^s V_n(z) dz \right)^{-\frac{1}{p}} d_s \left(\int_0^s V_n(z) dz \right)
\end{aligned} \tag{4.44}$$

$$= p'\left(\int_0^s V_n(z)dz\right)^{\frac{1}{p'}}\Big|_0^t$$

$$\overset{\frac{1}{p'}>0}{=} p'\left(\int_0^t V_n(z)dz\right)^{\frac{1}{p'}}$$

$$= p'h^p(t).$$

By combining this fact in (4.43), we have

$$\int_{B(a,|x|_a)} f(y)dy \geq \left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{1}{p}} H_1^{\frac{1}{p'}}(r) \overset{(4.44)}{=} (p')^{\frac{1}{p'}}\left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{1}{p}} h^{\frac{p}{p'}}(r), \tag{4.45}$$

by multiplying u, integrating over $\mathbb{X}$ with $q<0$ and by using (direct) Minkowski's inequality with $\frac{q}{p} \geq 1$, we compute

$$\begin{aligned}
&\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx \\
&= \int_0^\infty \int_{\sum_\omega} u(r,\omega)\lambda(r,\omega)\left(\int_0^r\int_{\sum_\sigma}\lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q drd\omega \\
&\overset{(4.42)}{=} \int_0^\infty U_1(r)\left(\int_0^r\int_{\sum_\sigma}\lambda(s,\sigma)f(s,\sigma)dsd\sigma\right)^q dr \\
&\overset{q<0,\,(4.45)}{\leq} (p')^{\frac{q}{p'}}\int_0^\infty U_1(r)\left(\int_0^r h^p(s)F_n(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(r)dr \\
&= (p')^{\frac{q}{p'}}\int_0^\infty U_1(r)\left(\int_0^\infty \chi_{\{0,r\}}h^p(s)F_n(s)ds\right)^{\frac{q}{p}} h^{\frac{qp}{p'}}(r)dr \\
&\leq (p')^{\frac{q}{p'}}\left[\int_0^\infty h^p(s)F_n(s)\left(\int_s^\infty U_1(r)h^{\frac{qp}{p'}}(r)dr\right)^{\frac{p}{q}}ds\right]^{\frac{q}{p}},
\end{aligned} \tag{4.46}$$

where $\chi_{\{0,r\}}$ is the cut-off function. At the same time, we can also estimate

$$\begin{aligned}
h^{\frac{pq}{p'}}(t) &= \left[\left(\int_0^t\int_{\sum_\sigma}\lambda(s,\sigma)v^{1-p'}(s,\sigma)dsd\sigma\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}} \\
&\overset{(4.39)}{=} \left[\left(\int_0^t V_n(s)ds\right)^{\frac{q}{p'}}\right]^{\frac{1}{p'}} \\
&= \left[\left(\int_0^t V_n(s)ds\right)^{\frac{q}{p'}}\left(\int_0^t U_1(s)ds\right)\left(\int_0^t U_1(s)ds\right)^{-1}\right]^{\frac{1}{p'}} \\
&= \mathcal{D}_1^{\frac{q}{p'}}(|t|_a)\left(\int_0^t U_1(s)ds\right)^{-\frac{1}{p'}},
\end{aligned} \tag{4.47}$$

where $\mathcal{D}_1(|t|_a) := \left(\int_0^t V_n(s)ds\right)^{\frac{1}{p'}} \left(\int_0^t U_1(s)ds\right)^{\frac{1}{q}}$. By using this fact and non-decreasing of $\mathcal{D}_1(|x|_a)$, we get

$$\begin{aligned}
&\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx \\
&\overset{(4.46)}{\leq} (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\left(\int_s^\infty U_1(r)h^{\frac{qp}{p'}}(r)dr\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{\frac{p}{q}>0,\,(4.47)}{\leq} (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s)\left(\int_s^\infty U_1(r)\left(\int_0^r U_1(z)dz\right)^{-\frac{1}{p'}} dr\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s)\left(\int_s^\infty d_r\left[p\left(\int_0^r U_1(z)dz\right)^{\frac{1}{p}}\right]\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&= (p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s)\left(p\left(\int_0^\infty U_1(z)dz\right)^{\frac{1}{p}} - p\left(\int_0^s U_1(z)dz\right)^{\frac{1}{p}}\right)^{\frac{p}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty h^p(s)F_n(s)\mathcal{D}_1^{\frac{p}{p'}}(s)\left(\int_0^s U_1(z)dz\right)^{\frac{1}{q}} ds\right]^{\frac{q}{p}} \\
&\overset{(4.40)}{=} (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F_n(s)\mathcal{D}_1^{1+\frac{p}{p'}}(s)ds\right]^{\frac{q}{p}} \\
&= (-p)(p')^{\frac{q}{p'}} \left[\int_0^\infty F_n(s)\mathcal{D}_1^{p}(s)ds\right]^{\frac{q}{p}} \\
&\overset{p<0}{\leq} (-p)(p')^{\frac{q}{p'}} D_1^q \left[\int_0^\infty F_n(s)ds\right]^{\frac{q}{p}} \\
&\overset{(4.38)}{=} (-p)(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{q}{p}} \\
&= |p|(p')^{\frac{q}{p'}} D_1^q \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{q}{p}}.
\end{aligned} \tag{4.48}$$

Finally,

$$\left(\int_{\mathbb{X}} \left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right)^{\frac{1}{q}} \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1 \left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}}. \tag{4.49}$$

Hence, it follows that (4.35) holds with $C_1(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}} D_1$ proving one of the relations in (4.37).

Step 2. In this step we show the biggest constant satisfies $C_1(p,q) \leq D_1$. Let us denote by $f(x) = v^{1-p'}\chi_{\{0,t\}}(|x|_a)$. Then we have

$$
\begin{aligned}
\left[\int_{\mathbb{X}}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} &\overset{\frac{1}{q}<0}{\leq} \left[\int_{|x|_a\geq t}\left(\int_{B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \\
&= \left[\int_{|x|_a\geq t}\left(\int_{|y|_a\leq t} v^{1-p'}(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \qquad (4.50) \\
&= \left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]\left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}},
\end{aligned}
$$

and

$$
C_1(p,q)\left[\int_{\mathbb{X}} f^p(x)v(x)dx\right]^{\frac{1}{p}} = C_1(p,q)\left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]^{\frac{1}{p}}. \qquad (4.51)
$$

By using above facts, we obtain

$$
C_1(p,q) \leq \left[\int_{|y|_a\leq t} v^{1-p'}(y)dy\right]^{\frac{1}{p'}}\left[\int_{|x|_a\geq t} u(x)dx\right]^{\frac{1}{q}}. \qquad (4.52)
$$

Finally, we get $C_1(p,q) \leq D_1$. ☐

Then let us give conjugate integral Hardy inequality.

Theorem 4.7. *Assume that $p, q < 0$ such that $q \leq p < 0$. Let $\mathbb{X}$ be a metric measure space with a polar decomposition at $a \in \mathbb{X}$. Suppose that $u, v \geq 0$ are locally integrable functions on $\mathbb{X}$. Then the inequality*

$$
\left[\int_{\mathbb{X}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} f(y)dy\right)^q u(x)dx\right]^{\frac{1}{q}} \geq C_2(p,q)\left(\int_{\mathbb{X}} f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad (4.53)
$$

holds for all non-negative real-valued measurable functions f, if

$$
0 < D_2 = \inf_{x\neq a}\mathcal{D}_2(|x|_a) = \inf_{x\neq a}\left[\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{X}\setminus B(a,|x|_a)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\right], \qquad (4.54)
$$

and $\mathcal{D}_2(|x|_a)$ is a non-increasing. Moreover, biggest constant C satisfies

$$
D_2 \geq C_2(p,q) \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}D_2, \qquad (4.55)
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. The main idea proof of this theorem similar with Theorem 4.6, except we should use $\mathcal{D}_2(|x|_a)$ is a non-increasing. ☐

4.3. **Reverse Hardy inequality with $q < 0$ and $p \in (0,1)$ on the homogeneous Lie groups.** Then we have the following reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 4.8. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Suppose that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{B(0,|x|)}f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}}\geq C\left(\int_{\mathbb{G}}f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.56}$$

holds for $C>0$ and for all non-negative measurable functions f, if only if

$$\alpha+Q<0,\ \ \beta(1-p')+Q>0\ \ and\ \ \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{4.57}$$

Moreover, the biggest constant C for (4.56) satisfies

$$\begin{aligned}&\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\geq C\\ &\geq\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}},\end{aligned} \tag{4.58}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Let us verify condition (4.10) with $u(x)=|x|^{\alpha}$, $v(x)=|x|^{\beta}$ and with $a=0$. By calculating the first integral in (4.10), we obtain

$$\begin{aligned}\int_{\mathbb{G}\backslash B(0,|x|)}u(y)dy&=\int_{\mathbb{G}\backslash B(0,|x|)}|y|^{\alpha}dy\overset{(2.11)}{=}\int_{|x|}^{\infty}\int_{\mathfrak{S}}\rho^{\alpha}\rho^{Q-1}d\rho d\sigma(\omega)\\ &=|\mathfrak{S}|\int_{|x|}^{\infty}\rho^{Q+\alpha-1}d\rho\overset{Q+\alpha<0}{=}-\frac{|\mathfrak{S}|}{Q+\alpha}|x|^{Q+\alpha}=\frac{|\mathfrak{S}|}{|Q+\alpha|}|x|^{Q+\alpha},\end{aligned} \tag{4.59}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned}\int_{B(0,|x|)}v^{1-p'}(y)dy&=\int_{B(0,|x|)}|y|^{\beta(1-p')}dy\overset{(2.11)}{=}\int_{0}^{|x|}\int_{\mathfrak{S}}\rho^{\beta(1-p')}\rho^{Q-1}d\rho d\sigma(\omega)\\ &=|\mathfrak{S}|\int_{0}^{|x|}\rho^{Q+\beta(1-p')-1}d\rho\\ &\overset{Q+\beta(1-p')>0}{=}\frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}.\end{aligned} \tag{4.60}$$

Finally by summarising above facts with $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$, we get

$$\begin{aligned}D_1&=\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\inf_{r>0}r^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}}\\ &=\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}>0.\end{aligned} \tag{4.61}$$

From (4.11), we obtain

$$\begin{aligned} &\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C \\ &\geq \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}, \end{aligned} \tag{4.62}$$

completing the proof. $\square$

Similarly, we have conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Corollary 4.9. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the conjugate reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.63}$$

holds for $C>0$ and for all non-negative measurable functions f, if only if

$$\alpha+Q>0,\ \ \beta(1-p')+Q<0 \ \text{ and } \ \frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0. \tag{4.64}$$

Moreover, the biggest constant C for (4.63) satisfies

$$\begin{aligned} &\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}} \geq C \\ &\geq \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}\left(\frac{p'}{p'+q}\right)^{-\frac{1}{q}}\left(\frac{q}{p'+q}\right)^{-\frac{1}{p'}}, \end{aligned} \tag{4.65}$$

where $|\mathfrak{S}|$ is the area of unit sphere with respect to $|\cdot|$.

Proof. Proof of this corollary is similar to the previous case. $\square$

4.4. **Reverse Hardy inequality with $\infty<q\leq p<0$ on the homogeneous Lie groups.** In this section we show the reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 4.10. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q\leq p<0$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{G}}\left(\int_{B(0,|x|)} f(y)dy\right)^{q}|x|^{\alpha}dx\right]^{\frac{1}{q}} \geq C_1\left(\int_{\mathbb{G}} f^{p}(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.66}$$

holds for $C_1>0$ and for all non-negative measurable functions f, if $\alpha+Q>0$, $\beta(1-p')+Q>0$ and $\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}=0$. Moreover, the biggest constant C_1 for (4.66) satisfies

$$\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \geq C_1 \geq |p|^{\frac{1}{q}}(p')^{\frac{1}{p'}}\left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}.$$

Proof. Let us check verify (4.36) with $u(x) = |x|^{\alpha}$ and $v(x) = |x|^{\beta}$. Let us calculate the first integral in (4.36):

$$\begin{aligned}\int_{B(0,|x|)} u(y)dy = \int_{B(0,|x|)} |y|^{\alpha} dy &\overset{(2.11)}{=} \int_0^{|x|}\int_{\mathfrak{S}} r^{\alpha} r^{Q-1} dr d\sigma \\ &= |\mathfrak{S}| \int_0^{|x|} r^{Q+\alpha-1} dr \overset{Q+\alpha>0}{=} \frac{|\mathfrak{S}|}{Q+\alpha}|x|^{Q+\alpha},\end{aligned} \tag{4.67}$$

where $|\mathfrak{S}|$ is the area of the unit quasi-sphere in $\mathbb{G}$. Then,

$$\begin{aligned}\int_{B(0,|x|)} v^{1-p'}(y)dy = \int_{B(0,|x|)} |y|^{\beta(1-p')} dy &\overset{(2.11)}{=} \int_0^{|x|}\int_{\mathfrak{S}} r^{\beta(1-p')} r^{Q-1} dr d\sigma \\ &= |\mathfrak{S}| \int_0^{|x|} r^{Q+\beta(1-p')-1} dr \overset{Q+\beta(1-p')>0}{=} \frac{|\mathfrak{S}|}{Q+\beta(1-p')}|x|^{Q+\beta(1-p')}.\end{aligned} \tag{4.68}$$

Finally by summarising above facts with $\frac{Q+\alpha}{q} + \frac{Q+\beta(1-p')}{p'} = 0$, we have

$$\begin{aligned}\mathcal{D}_1(|x|) &= \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} \left[|x|^{\frac{Q+\alpha}{q}+\frac{Q+\beta(1-p')}{p'}}\right] \\ &= \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.69}$$

it means $\mathcal{D}_1(|x|)$ is a non-decreasing function. Then

$$D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}} > 0.$$

Therefore, by (4.37) we have

$$D_1 \geq C_1 \geq |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} D_1,$$

where $D_1 = \left(\frac{|\mathfrak{S}|}{\alpha+Q}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{Q+\beta(1-p')}\right)^{\frac{1}{p'}}$, completing the proof. □

Then we have conjugate reverse integral Hardy inequality on homogeneous Lie groups.

Theorem 4.11. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q with a quasi-norm $|\cdot|$. Assume that $q \leq p < 0$ and $\alpha, \beta \in \mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$\left[\int_{\mathbb{G}} \left(\int_{\mathbb{G}\setminus B(0,|x|)} f(y)dy\right)^q |x|^{\alpha} dx\right]^{\frac{1}{q}} \geq C_2 \left(\int_{\mathbb{G}} f^p(x)|x|^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.70}$$

holds for $C_2 > 0$ and for all non-negative measurable functions f, if $\alpha + Q < 0$, $\beta(1-p') + Q < 0$ and $\frac{Q+\alpha}{q} + \frac{Q+\beta(1-p')}{p'} = 0$. Moreover, the biggest constant C_2 for (4.70) satisfies

$$\left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}} \geq C_2 \geq |p|^{\frac{1}{q}} (p')^{\frac{1}{p'}} \left(\frac{|\mathfrak{S}|}{|\alpha+Q|}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{|Q+\beta(1-p')|}\right)^{\frac{1}{p'}}.$$

Proof. Proof of this theorem is similar to the previous case, but we need using Theorem 4.7. ☐

4.5. **Reverse Hardy inequality with $q<0$ and $p\in(0,1)$ on the hyperbolic space.** Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a\in\mathbb{H}^n$. Let us set

$$u(x)=(\sinh|x|_a)^{\alpha},\quad v(x)=(\sinh|x|_a)^{\beta}. \tag{4.71}$$

Then we have the following result of this subsection.

Corollary 4.12. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and let $a\in\mathbb{H}^n$. Assume that $q<0$, $p\in(0,1)$ and $\alpha,\beta\in\mathbb{R}$. Then the reverse integral Hardy inequality*

$$\left[\int_{\mathbb{H}^n}\left(\int_{B(a,|x|_a)}f(y)dy\right)^q(\sinh|x|_a)^{\alpha}dx\right]^{\frac{1}{q}}\geq C\left(\int_{\mathbb{H}^n}f^p(x)(\sinh|x|_a)^{\beta}dx\right)^{\frac{1}{p}}, \tag{4.72}$$

holds for $C>0$ and for all non-negative measurable functions f, if

$$0\leq\alpha+n<1,\;\;\beta(1-p')+n>0\;\;\text{and}\;\;\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}. \tag{4.73}$$

Proof. Let us verify condition (4.10). By using polar decomposition for the hyperbolic space, we have

$$D_1=\inf_{x\neq a}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}. \tag{4.74}$$

If $\alpha+n<1$ and $\beta(1-p')+n>0$, then (4.74) is integrable. Let us check the finiteness and positiveness of the infimum (4.74). Let us divide the proof in two cases.

First case, $|x|_a\gg1$. Then $\sinh|x|_a\approx\exp|x|_a$ if $|x|_a\gg1$. Then we obtain,

$$\begin{aligned}D_1^1&=\inf_{|x|_a\gg1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\&\simeq\inf_{|x|_a\gg1}\left(\int_{|x|_a}^{\infty}(\exp\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\&=\inf_{|x|_a\gg1}\left((\exp|x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}}\left((\exp|x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}}\\&=\inf_{|x|_a\gg1}(\exp|x|_a)^{\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}},\end{aligned} \tag{4.75}$$

infimum of the last term is positive, if only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}\geq0$, i.e., $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}$, then $D_1^1>0$.

Let us consider the second case $|x|_a \ll 1$. For $|x|_a \ll 1$ we have $\sinh \rho_{\{0\leq\rho<|x|_a\}} \approx \rho$, then we calculate

$$
\begin{aligned}
&\inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\sinh \rho)^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\quad \simeq \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{R} (\sinh \rho)^{\alpha+n-1} d\rho + \int_R^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_0^{|x|_a} \rho^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\quad \simeq \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{R} (\sinh \rho)^{\alpha+n-1} d\rho + \int_R^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} |x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned}
\tag{4.76}
$$

Similarly, for small R we have $\sinh \rho_{\{|x|_a\leq\rho<R\}} \approx \rho$, so that we obtain

$$
\begin{aligned}
&\inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\sinh \rho)^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\quad \simeq \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{R} (\sinh \rho)^{\alpha+n-1} d\rho + \int_R^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} |x|_a^{\frac{\beta(1-p')+n}{p'}} \\
&\quad \simeq \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{R} \rho^{\alpha+n-1} d\rho + \int_R^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} |x|_a^{\frac{\beta(1-p')+n}{p'}} \\
&\quad \simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\alpha+n} + C_R \right)^{\frac{1}{q}} |x|_a^{\frac{\beta(1-p')+n}{p'}}.
\end{aligned}
\tag{4.77}
$$

If $\alpha + n \geq 0$, we have $\frac{\alpha+n}{q} \leq 0$, then we get

$$
\begin{aligned}
D_1^2 &= \inf_{|x|_a \ll 1} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_0^{|x|_a} (\sinh \rho)^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\alpha+n} + C_R \right)^{\frac{1}{q}} |x|_a^{\frac{\beta(1-p')+n}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} |x|_a^{\frac{\beta(1-p')+n}{p'}} > 0,
\end{aligned}
\tag{4.78}
$$

and infimum is positive, if only if $\frac{\beta(1-p')+n}{p'} < 0$, i.e., $\beta(1-p') + n > 0$. $\square$

Let us give the reverse conjugate integral Hardy's inequality in hyperbolic spaces:

Corollary 4.13. *Let $\mathbb{H}^n$ be the hyperbolic space of dimension n and $a \in \mathbb{H}^n$. Assume that $q < 0$, $p \in (0,1)$ and let $\alpha, \beta \in \mathbb{R}$. Then the reverse conjugate integral Hardy inequality*

$$
\left[\int_{\mathbb{H}^n} \left(\int_{\mathbb{X} \setminus B(a,|x|_a)} f(y) dy \right)^q (\sinh |x|_a)^{\alpha} dx \right]^{\frac{1}{q}} \geq C \left(\int_{\mathbb{H}^n} f^p(x) (\sinh |x|_a)^{\beta} dx \right)^{\frac{1}{p}},
\tag{4.79}
$$

holds for all non-negative measurable functions f, if

$$\alpha+n>0,\ \ 1>\beta(1-p')+n\geq 0 \ \ \textit{and}\ \ \frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq \frac{1}{q}+\frac{1}{p'}.$$

Proof. Similarly to the previous case, check condition (4.26) and then, we have

$$D_2=\inf_{x\neq a}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}. \tag{4.80}$$

If $\alpha+n>0$ and $\beta(1-p')+n<1$, then (4.80) is integrable. If $|x|_a\gg 1$, we obtain,

$$\begin{aligned}
D_2^1&=\inf_{|x|_a\gg 1}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq\inf_{|x|_a\gg 1}\left(\int_{|x|_a}^{\infty}(\exp\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_0^{|x|_a}(\exp\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&=\inf_{|x|_a\gg 1}\left((\exp|x|_a)^{\alpha+n-1}\right)^{\frac{1}{q}}\left((\exp|x|_a)^{\beta(1-p')+n-1}\right)^{\frac{1}{p'}}\\
&=\inf_{|x|_a\gg 1}(\exp|x|_a)^{\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}},
\end{aligned} \tag{4.81}$$

infimum of the last term is positive, if only if $\frac{\alpha+n-1}{q}+\frac{\beta(1-p')+n-1}{p'}\geq 0$, i.e., $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}$, then $D_1^1>0$.

If $|x|_a\ll 1$, we obtain

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq\inf_{|x|_a\ll 1}\left(\int_0^{|x|_a}\rho^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}|x|_a^{\frac{\alpha+n}{q}}.
\end{aligned} \tag{4.82}$$

Similarly, for small R we have $\sinh\rho_{\{|x|_a\leq\rho<R\}}\approx\rho$, so that we obtain

$$\begin{aligned}
&\inf_{|x|_a\ll 1}\left(\int_0^{|x|_a}(\sinh\rho)^{\alpha+n-1}d\rho\right)^{\frac{1}{q}}\left(\int_{|x|_a}^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}\\
&\simeq\inf_{|x|_a\ll 1}\left(\int_{|x|_a}^{R}(\sinh\rho)^{\beta(1-p')+n-1}d\rho+\int_R^{\infty}(\sinh\rho)^{\beta(1-p')+n-1}d\rho\right)^{\frac{1}{p'}}|x|_a^{\frac{\alpha+n}{q}}\\
&\simeq\inf_{|x|_a\ll 1}\left(|x|_a^{\beta(1-p')+n}+C_R'\right)^{\frac{1}{q}}|x|_a^{\frac{\alpha+n}{q}}.
\end{aligned} \tag{4.83}$$

If $\beta(1-p')+n \geq 0$, we have $\frac{\beta(1-p')+n}{q} \leq 0$, then we have

$$
\begin{aligned}
D_2^2 &= \inf_{|x|_a \ll 1} \left(\int_0^{|x|_a} (\sinh \rho)^{\alpha+n-1} d\rho \right)^{\frac{1}{q}} \left(\int_{|x|_a}^{\infty} (\sinh \rho)^{\beta(1-p')+n-1} d\rho \right)^{\frac{1}{p'}} \\
&\simeq \inf_{|x|_a \ll 1} \left(|x|_a^{\beta(1-p')+n} + C_R' \right)^{\frac{1}{q}} |x|_a^{\frac{\alpha+n}{q}} \\
&\simeq \inf_{|x|_a \ll 1} |x|_a^{\frac{\alpha+n}{q}},
\end{aligned} \tag{4.84}
$$

and infimum is positive, if only if $\frac{\alpha+n}{q} < 0$, i.e., $\alpha+n > 0$. $\square$

4.6. **Reverse Hardy inequality with $q < 0$ and $p \in (0,1)$ on the Cartan-Hadamard manifolds.** Let (M, g) be the Cartan-Hadamard manifold with curvature K_M. If $K_M = 0$ then $J(t, \omega) = 1$ and we set

$$
u(x) = |x|_a^{\alpha}, \quad v(x) = |x|_a^{\beta}, \text{ when } K_M = 0. \tag{4.85}
$$

If $K_M < 0$ then $J(t, \omega) = \left(\frac{\sinh \sqrt{b}t}{\sqrt{b}t} \right)^{n-1}$ and we set

$$
u(x) = (\sinh \sqrt{-K_M}|x|_a)^{\alpha}, \quad v(x) = (\sinh \sqrt{-K_M}|x|_a)^{\beta}, \text{ when } K_M < 0. \tag{4.86}
$$

Then we have the following result of this subsection.

Corollary 4.14. *Assume that (M, g) be the Cartan-Hadamard manifold of dimension n and with curvature K_M. Assume that $q < 0$, $p \in (0,1)$ and $\alpha, \beta \in \mathbb{R}$. Then we have*

i) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$
\left[\int_M \left(\int_{B(a,|x|_a)} f(y) dy \right)^q |x|_a^{\alpha} dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x) |x|^{\beta} dx \right)^{\frac{1}{p}}, \tag{4.87}
$$

holds for $C > 0$ and for non-negative measurable functions f, if only if $\alpha+n < 0$, $\beta(1-p')+n > 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

ii) *if $K_M = 0$, $u(x) = |x|_a^{\alpha}, v(x) = |x|_a^{\beta}$, then*

$$
\left[\int_M \left(\int_{M \setminus B(a,|x|_a)} f(y) dy \right)^q |x|^{\alpha} dx \right]^{\frac{1}{q}} \geq C \left(\int_M f^p(x) |x|^{\beta} dx \right)^{\frac{1}{p}}, \tag{4.88}
$$

holds for $C > 0$ and for non-negative measurable functions f, if only if $\alpha+n > 0$, $\beta(1-p')+n < 0$ and $\frac{n+\alpha}{q} + \frac{n+\beta(1-p')}{p'} = 0$;

iii) *if $K_M < 0$, $u(x) = (\sinh \sqrt{-K_M}|x|_a)^{\alpha}, v(x) = (\sinh |x|_a)^{\beta}$, then*

$$
\left[\int_M \left(\int_{B(a,|x|_a)} f(y) dy \right)^q (\sinh \sqrt{-K_M}|x|_a)^{\alpha} dx \right]^{\frac{1}{q}}
$$
$$
\geq C \left(\int_M f^p(x) (\sinh \sqrt{-K_M}|x|_a)^{\beta} dx \right)^{\frac{1}{p}}, \tag{4.89}
$$

holds for $C > 0$ and for all non-negative measurable functions f, if $0 \leq \alpha+n < 1$, $\beta(1-p')+n > 0$ and $\frac{\alpha+n}{q} + \frac{\beta(1-p')+n}{p'} \geq \frac{1}{q} + \frac{1}{p'}$;

iv) *if* $K_M<0$, $u(x)=(\sinh\sqrt{-K_M}|x|_a)^{\alpha}$, $v(x)=(\sinh\sqrt{-K_M}|x|_a)^{\beta}$, *then*

$$\left[\int_M\left(\int_{M\setminus B(a,|x|_a)}f(y)dy\right)^q(\sinh\sqrt{-K_M}|x|_a)^{\alpha}dx\right]^{\frac{1}{q}}$$
$$\geq C\left(\int_M f^p(x)(\sinh\sqrt{-K_M}|x|_a)^{\beta}dx\right)^{\frac{1}{p}},\quad(4.90)$$

holds for $C>0$ *and for all non-negative measurable functions* f, *if* $\alpha+n>0$, $1>\beta(1-p')+n\geq 0$ *and* $\frac{\alpha+n}{q}+\frac{\beta(1-p')+n}{p'}\geq\frac{1}{q}+\frac{1}{p'}$.

4.7. **Reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities with $q<0$ and $p\in(0,1)$ on the homogeneous Lie groups.** Now we formulate the reverse Stein-Weiss inequality on homogeneous Lie group.

Theorem 4.15. *Let* $\mathbb{G}$ *be a homogeneous group of homogeneous dimension* $Q\geq 1$ *and let* $|\cdot|$ *be an arbitrary homogeneous quasi-norm on* $\mathbb{G}$. *Assume that* $\lambda>0$, $p,q'\in(0,1)$, $0\leq\alpha<-\frac{Q}{q}$, $0\leq\beta<-\frac{Q}{p'}$, $\frac{1}{q'}+\frac{1}{p}=\frac{\alpha+\beta+\lambda}{Q}+2$, *where* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$. *Then for all non-negative functions* $f\in L^{q'}(\mathbb{G})$ *and* $h\in L^p(\mathbb{G})$ *we have*

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}f(x)h(y)|y|^{\beta}dxdy\geq C\|f\|_{L^{q'}(\mathbb{G})}\|h\|_{L^p(\mathbb{G})},\quad(4.91)$$

where C *is a positive constant independent of* f *and* h.

Proof. By using reverse Hölder's inequality with $\frac{1}{q}+\frac{1}{q'}=1$ (Theorem 4.1) in (4.91), we calculate,

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^{\alpha}f(x)|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dydx=\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dy\right)f(x)dx$$
$$\stackrel{(4.1)}{\geq}\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dy\right)^q dx\right)^{\frac{1}{q}}\|f\|_{L^{q'}(\mathbb{G})}.$$

For (4.91), it is enough to show that

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dy\right)^q dx\right)^{\frac{1}{q}}\geq C\|h\|_{L^p(\mathbb{G})},$$

and by changing $u(y)=h(y)|y|^{\beta}$, this is equivalent to

$$\int_{\mathbb{G}}\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q dx\leq C\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.$$

We have that

$$\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\geq\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy,$$

then

$$\left(\int_{\mathbb{G}}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q\stackrel{q<0}{\leq}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q.$$

Hence, we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}$$
$$\stackrel{q<0}{\geq}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}:=I_1^{\frac{1}{q}}. \quad (4.92)$$

Similarly with (4.92), we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}$$
$$\stackrel{q<0}{\geq}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}:=I_2^{\frac{1}{q}}. \quad (4.93)$$

By summarising above facts, from (4.92)-(4.93), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}. \quad (4.94)$$

From now on, in view of Proposition 2.6 we can assume that our quasi-norm is actually a norm.

Step 1. Let us consider I_1. From Proposition 2.6 and the properties of the quasi-norm with $|y|\leq\frac{|x|}{2}$, we have

$$|x|=|x^{-1}|=|x^{-1}yy^{-1}|\leq|x^{-1}y|+|y^{-1}|=|y^{-1}x|+|y|\leq|y^{-1}x|+\frac{|x|}{2}. \quad (4.95)$$

For any $\lambda>0$, we have

$$2^{-\lambda}|x|^{\lambda}\leq|y^{-1}x|^{\lambda}.$$

It means,

$$2^{-\lambda}\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}u(y)dy\leq\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy,$$

so that

$$\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}\leq 2^{-\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\lambda}u(y)dy\right)^{q}.$$

Therefore, we have

$$\begin{aligned}I_1&=\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\\&\leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}dx.\end{aligned}$$

Assume that $W(x)=|x|^{(\alpha+\lambda)q}$ and $U(y)=|y|^{-\beta p}$, if condition (4.10) in Theorem 4.4 is satisfied, then by (4.9) we have

$$I_1 \leq 2^{-\lambda q}\int_{\mathbb{G}}\left(\int_{B(0,\frac{|x|}{2})}u(y)dy\right)^q |x|^{(\alpha+\lambda)q}dx \leq C_1\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.$$

Let us check condition (3.136). From assumption $\beta<-\frac{Q}{p'}$, we get

$$\frac{1}{p}+\frac{1}{q'}=\frac{\alpha+\beta+\lambda}{Q}+2<\frac{\alpha+\lambda}{Q}-\frac{1}{p'}+2,$$

that is, $\frac{Q+(\alpha+\lambda)q}{Qq}>0$, then $Q+(\alpha+\lambda)q<0$ and by using the polar decomposition (2.11):

$$\begin{aligned}\left(\int_{\mathbb{G}\setminus B(0,|x|)}W(y)dy\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G}\setminus B(0,|x|)}|y|^{(\alpha+\lambda)q}dy\right)^{\frac{1}{q}}\\ &= \left(\int_{|x|}^{\infty}\int_{\mathfrak{S}}r^{Q-1}r^{(\alpha+\lambda)q}drd\sigma(\omega)\right)^{\frac{1}{q}}\\ &= \left(|\mathfrak{S}|\int_{|x|}^{\infty}r^{Q-1+(\alpha+\lambda)q}dr\right)^{\frac{1}{q}}\\ &= \left(-\frac{|\mathfrak{S}|}{Q+(\alpha+\lambda)q}|x|^{Q+(\alpha+\lambda)q}\right)^{\frac{1}{q}}\\ &= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}|x|^{\frac{Q+(\alpha+\lambda)q}{q}}.\end{aligned}$$

From $\beta<-\frac{Q}{p'}$, we get

$$-\beta p(1-p')+Q>-\beta p(1-p')-\beta p'=0.$$

It means $-\beta p(1-p')+Q>0$. Let us consider

$$\begin{aligned}\left(\int_{B(0,|x|)}U^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{B(0,|x|)}|y|^{-\beta p(1-p')}dy\right)^{\frac{1}{p'}}\\ &= \left(\int_0^{|x|}\int_{\mathfrak{S}}r^{-\beta p(1-p')}r^{Q-1}drd\sigma(\omega)\right)^{\frac{1}{p'}}\\ &= \left(|\mathfrak{S}|\int_0^{|x|}r^{-\beta p(1-p')+Q-1}dr\right)^{\frac{1}{p'}}\\ &= \left(\frac{|\mathfrak{S}|}{-\beta p(1-p')+Q}|x|^{-\beta p(1-p')+Q}\right)^{\frac{1}{p'}}\\ &= \left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}}|x|^{\frac{-\beta p(1-p')+Q}{p'}}.\end{aligned} \tag{4.96}$$

Hence, we get

$$
\begin{aligned}
A_1 &= \inf_{x\neq a}\left(\int_{\mathbb{G}\setminus B(0,|x|)} W(y)dx\right)^{\frac{1}{q}}\left(\int_{B(0,|x|)} U^{1-p'}(y)dy\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}} \inf_{x\neq a}|x|^{\frac{(\alpha+\lambda)q+Q}{q}+\frac{-\beta p(1-p')+Q}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}} \inf_{x\neq a}|x|^{Q\left(\frac{1}{q}+\frac{1}{p'}+\frac{\alpha+\beta+\lambda}{Q}\right)} \qquad (4.97)\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}} \inf_{x\neq a}|x|^{Q\left(2-\frac{1}{q'}-\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\alpha+\lambda)q|}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{Q-\beta p(1-p')}\right)^{\frac{1}{p'}} > 0.
\end{aligned}
$$

From (4.9), we have

$$
I_1 \leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^q dx \leq 2^{-\lambda q}C_1^q\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}, \qquad (4.98)
$$

so that

$$
I_1^{\frac{1}{q}} \geq 2^{-\lambda}C_1\||y|^{-\beta}u\|_{L^p(\mathbb{G})} = 2^{-\lambda}C_1\|h\|_{L^p(\mathbb{G})}. \qquad (4.99)
$$

Step 2. As in the previous case I_1, now we consider I_2. From $2|x|\leq|y|$, we calculate

$$
|y| = |y^{-1}| = |y^{-1}xx^{-1}| \leq |y^{-1}x| + |x| \leq |y^{-1}x| + \frac{|y|}{2},
$$

that is,

$$
\frac{|y|}{2} \leq |y^{-1}x|.
$$

Assume that $W(x)=|x|^{\alpha q}$ and $U(y)=|y|^{-(\beta+\lambda)p}$ and if condition (4.26) is satisfied, then we have

$$
\begin{aligned}
I_2 &= \int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} |x|^{\alpha}|y^{-1}x|^{\lambda}u(y)dy\right)^q dx\\
&\leq 2^{-\lambda q}\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)|y|^{\lambda}dy\right)^q dx \leq 2^{-\lambda q}\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.
\end{aligned}
$$

Let us verify condition (4.26). Then we get

$$
\begin{aligned}
\left(\int_{B(0,|x|)} W(y)dy\right)^{\frac{1}{q}} &= \left(\int_{B(0,|x|)} |y|^{\alpha q}dy\right)^{\frac{1}{q}}\\
&= \left(\int_0^{|x|}\int_{\mathfrak{S}} r^{\alpha q}r^{Q-1}drd\sigma(\omega)\right)^{\frac{1}{q}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}|x|^{\frac{Q+\alpha q}{q}},
\end{aligned}
$$

where $Q+\alpha q>0$. By using $\alpha<-\frac{Q}{q}$, we get

$$\frac{1}{q'}+\frac{1}{p}=\frac{\alpha+\beta+\lambda}{Q}+2<-\frac{1}{q}+\frac{\beta+\lambda}{Q}+2=\frac{\beta+\lambda}{Q}+1+\frac{1}{q'},$$

then

$$(\beta+\lambda)p'+Q<0. \tag{4.100}$$

By using this fact, we have

$$\begin{aligned}\left(\int_{\mathbb{G}\setminus B(0,|x|)}U^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{G}\setminus B(0,|x|)}|y|^{-(\beta+\lambda)(1-p')p}dy\right)^{\frac{1}{p'}}\\
&= \left(\int_{|x|}^{\infty}\int_{\mathfrak{S}}r^{Q-1}r^{-(\beta+\lambda)(1-p')p}drd\sigma(\omega)\right)^{\frac{1}{p'}}\\
&= \left(|\mathfrak{S}|\int_{|x|}^{\infty}r^{-(\beta+\lambda)(1-p')p+Q-1}dr\right)^{\frac{1}{p'}}\\
&\overset{(4.100)}{=} \left(-\frac{|\mathfrak{S}|}{Q-(\beta+\lambda)(1-p')p}|x|^{Q-(\beta+\lambda)(1-p')p}\right)^{\frac{1}{p'}}\\
&= \left(-\frac{|\mathfrak{S}|}{Q+(\beta+\lambda)p'}|x|^{Q+(\beta+\lambda)p'}\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}.\end{aligned}$$

Combining these facts we have

$$\begin{aligned}A_2 &= \inf_{x\neq a}\left(\int_{B(0,|x|)}W(y)dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\setminus B(0,|x|)}U^{1-p'}(y)dx\right)^{\frac{1}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{\frac{Q+\alpha q}{q}+\frac{Q+(\beta+\lambda)p'}{p'}}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{\frac{Q}{q}+\alpha+\frac{Q}{p'}+\beta+\lambda}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(\frac{1}{q}+\frac{1}{p'}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}\inf_{x\neq a}|x|^{Q\left(2-\frac{1}{q'}-\frac{1}{p}+\frac{\alpha+\beta+\lambda}{Q}\right)}\\
&= \left(\frac{|\mathfrak{S}|}{Q+\alpha q}\right)^{\frac{1}{q}}\left(\frac{|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}>0.\end{aligned} \tag{4.101}$$

Hence, we have

$$I_2=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^q dx\leq 2^{-\lambda q}C_2^q\||y|^{-\beta}u\|^q_{L^p(\mathbb{G})}.$$

Then, we have

$$I_2^{\frac{1}{q}} \geq 2^{-\lambda} C_2 \||y|^{-\beta} u\|_{L^p(\mathbb{G})} = 2^{-\lambda} C_2 \|h\|_{L^p(\mathbb{G})}. \tag{4.102}$$

Finally, from (4.99) and (4.102) in (4.94), we obtain

$$\begin{aligned}
\left(\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} u(y) dy\right)^q dx\right)^{\frac{1}{q}} &\geq \frac{I_1^{\frac{1}{q}}}{2} + \frac{I_2^{\frac{1}{q}}}{2} \\
&\geq \frac{2^{-\lambda}(C_1 + C_2)}{2} \||y|^{-\beta} u\|_{L^p(\mathbb{G})} \\
&= \frac{2^{-\lambda}(C_1 + C_2)}{2} \||y|^{-\beta} u\|_{L^p(\mathbb{G})} \\
&= C_3 \||y|^{-\beta} u\|_{L^p(\mathbb{G})},
\end{aligned} \tag{4.103}$$

where $C_3 = \frac{2^{-\lambda}(C_1+C_2)}{2} > 0$.

Theorem 4.15 is proved. □

Corollary 4.16. *By setting $\alpha = \beta = 0$ we get the reverse Hardy-Littlewood-Sobolev inequality on the homogeneous groups, in the following form:*

$$\int_{\mathbb{G}} \int_{\mathbb{G}} |y^{-1}x|^{\lambda} f(x) h(y) dx dy \geq C \|f\|_{L^{q'}(\mathbb{G})} \|h\|_{L^p(\mathbb{G})}, \tag{4.104}$$

for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $h \in L^p(\mathbb{G})$ with $\lambda > 0$, $p, q' \in (0,1)$, $\frac{1}{q'} + \frac{1}{p} = \frac{\lambda}{Q} + 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark 4.17. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, hence $Q = N$ and $|\cdot|$ can be any homogeneous quasi-norm on $\mathbb{R}^N$, in particular with the usual Euclidean distance, i.e. $|\cdot| = \|\cdot\|_E$, this was investigated in [54].*

Let us give improved reverse Stein-Weiss inequality.

Theorem 4.18. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq 1$ and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Suppose that $\lambda > 0$, $p, q' \in (0,1)$ and $\frac{1}{q'} + \frac{1}{p} = \frac{\alpha+\beta+\lambda}{Q} + 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $h \in L^p(\mathbb{G})$, inequality (4.91) holds, that is,*

$$\int_{\mathbb{G}} \int_{\mathbb{G}} |x|^{\alpha} |y^{-1}x|^{\lambda} f(x) h(y) |y|^{\beta} dx dy \geq C \|f\|_{L^{q'}(\mathbb{G})} \|h\|_{L^p(\mathbb{G})},$$

if one of the following conditions is satisfied:

(a) $0 \leq \alpha < -\frac{Q}{q}$.

(b) $0 \leq \beta < -\frac{Q}{p'}$.

Proof. Firstly, let us show (a). By using some notations from proof of Theorem 4.15 and (4.94), we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} u(y) dy\right)^q dx\right)^{\frac{1}{q}} \geq I_2^{\frac{1}{q}}, \tag{4.105}$$

and from Step 2 in the proof of Theorem 4.15 and by using (4.102), we get $I_2^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}$, then we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_2^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}. \tag{4.106}$$

Let us show (b). From(4.94), we get

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_1^{\frac{1}{q}}, \tag{4.107}$$

and from Step 1 in the proof of Theorem 4.15 and by using (4.92), we get $I_1^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}$, then we have

$$\left(\int_{\mathbb{G}} |x|^{\alpha q}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}u(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq I_1^{\frac{1}{q}} \geq C\||y|^{-\beta}u\|_{L^p(\mathbb{G})}. \tag{4.108}$$

□

4.8. **Reverse Hardy-Littlewood-Sobolev inequality with $-\infty < q < p < 0$ on the homogeneous Lie groups.** In this section, we prove reverse Hardy-Littlewood-Sobolev inequality and Stein-Weiss type inequality with $-\infty < q < p < 0$ on homogeneous Lie groups.

Let us present one of the main results of this section.

Theorem 4.19 (Reverse Hardy-Littlewood-Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension $Q \geq 1$ with a quasi-norm $|\cdot|$. Assume that $q < p < 0$, $\lambda < 0$ and $\frac{1}{p'} + \frac{1}{q} + \frac{\lambda}{Q} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $0 < \int_{\mathbb{G}} h^p(x)dx < \infty$,*

$$\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda}h(y)dxdy \geq C\left(\int_{\mathbb{G}} f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.109}$$

where C is a positive constant independent of f and h.

Proof. By using reverse Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$, we get

$$\begin{aligned}\int_{\mathbb{G}}\int_{\mathbb{G}} f(x)|y^{-1}x|^{\lambda}h(y)dydx &= \int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)f(x)dx \\ &\geq \left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)^q dx\right)^{\frac{1}{q}}\|f\|_{L^{q'}(\mathbb{G})}.\end{aligned}$$

So for (4.109), it is enough to show that

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}.$$

We have that

$$\int_{\mathbb{G}} |y^{-1}x|^{\lambda}h(y)dy \geq \int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\alpha}|y^{-1}x|^{\lambda}h(y)dy,$$

then

$$\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} h(y)dy\right)^q \overset{q<0}{\leq} \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y^{-1}x|^{\lambda} h(y)dy\right)^q.$$

Therefore, we obtain

$$\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} h(y)dy\right)^q dx\right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)} |y^{-1}x|^{\lambda} h(y)dy\right)^q dx\right)^{\frac{1}{q}}. \quad (4.110)$$

By using Proposition 2.7 with $|y| \leq \frac{|x|}{2}$, we get

$$|y^{-1}x| \overset{(2.10)}{\leq} C(|x|+|y|) \leq \frac{3C}{2}|x| = C_1|x|, \quad (4.111)$$

where $C>0$ and $C_1 = \frac{3C}{2}$. Then for any $\lambda<0$, we have

$$C_1^{\lambda}|x|^{\lambda} \leq |y^{-1}x|^{\lambda}.$$

It means,

$$C_1^{\lambda} \int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\lambda} h(y)dy \leq \int_{B\left(0,\frac{|x|}{2}\right)} |y^{-1}x|^{\lambda} h(y)dy,$$

so that

$$\left(\int_{B\left(0,\frac{|x|}{2}\right)} |y^{-1}x|^{\lambda} h(y)dy\right)^q \leq C_1^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\lambda} h(y)dy\right)^q.$$

Finally,

$$\begin{aligned}\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}} |y^{-1}x|^{\lambda} h(y)dy\right)^q dx\right)^{\frac{1}{q}} &\overset{q<0}{\geq} \left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)} |y^{-1}x|^{\lambda} h(y)dy\right)^q dx\right)^{\frac{1}{q}} \\ &\geq C_1^{\lambda}\left(\int_{\mathbb{G}} |x|^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)} h(y)dy\right)^q dx\right)^{\frac{1}{q}}.\end{aligned} \quad (4.112)$$

If condition (4.36) in Theorem 4.6 with $u(x)=|x|^{\lambda q}$ and $v(x)=1$ in (4.35) is satisfied, then we have

$$\left(\int_{\mathbb{G}} |x|^{\lambda q}\left(\int_{B\left(0,\frac{|x|}{2}\right)} h(y)dy\right)^q dx\right)^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}.$$

Let us start to check condition (4.36). From assumption, we have

$$0 = \frac{1}{p'}+\frac{1}{q}+\frac{\lambda}{Q} \overset{\frac{1}{p'}>0}{>} \frac{1}{q}+\frac{\lambda}{Q}, \quad (4.113)$$

it means $Q+\lambda q>0$. By using this fact, we obtain

$$\begin{aligned}\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy &= \int_{B\left(0,\frac{|x|}{2}\right)} |y|^{\lambda q}dy \overset{(2.11)}{=} \int_0^{\frac{|x|}{2}}\int_{\mathfrak{S}} r^{\lambda q} r^{Q-1} dr d\sigma \\ &= |\mathfrak{S}|\int_0^{\frac{|x|}{2}} r^{Q+\lambda q}dr \overset{Q+\lambda q>0}{=} \frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}|x|^{Q+\lambda q},\end{aligned} \quad (4.114)$$

and

$$\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy = \int_{B\left(0,\frac{|x|}{2}\right)} 1dy = \int_0^{\frac{|x|}{2}} \int_{\mathfrak{S}} r^{Q-1} dr d\sigma = \frac{|\mathfrak{S}|}{2^Q}|x|^Q. \tag{4.115}$$

Finally, by using assumption $\frac{1}{p'} + \frac{1}{q} + \frac{\lambda}{Q} = 0$,

$$\mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}} |x|^{\frac{Q}{p'}+\frac{Q+\lambda q}{q}} = \left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}}, \tag{4.116}$$

it means, $\mathcal{D}_1(|x|)$ is a non-decreasing function. Then,

$$D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{2^{Q+\lambda q}(Q+\lambda q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^Q}\right)^{\frac{1}{p'}} > 0.$$

□

Remark 4.20. *Inequality (4.109) is an even new in the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^n, +)$, $Q = n$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance).*

4.9. **Reverse Stein-Weiss type inequality with $-\infty < q \leq p < 0$ on the homogeneous Lie groups.** Let us show, the reverse Stein-Weiss type inequality on homogeneous Lie groups.

Theorem 4.21. *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension $Q \geq 1$ with any quasi-norm $|\cdot|$. Assume that $q \leq p < 0$, $\lambda < 0$, $\beta > -\frac{Q}{p'}$, $\alpha > -\frac{Q}{q}$ and $\frac{1}{p'} + \frac{1}{q} + \frac{\alpha+\beta+\lambda}{Q} = 0$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(\mathbb{G})$ and $0 < \int_{\mathbb{G}} h^p(x)dx < \infty$,*

$$\int_{\mathbb{G}}\int_{\mathbb{G}} |x|^{\alpha} f(x) |y^{-1}x|^{\lambda} h(y) |y|^{\beta} dxdy \geq C \left(\int_{\mathbb{G}} f^{q'}(x)dx\right)^{\frac{1}{q'}} \left(\int_{\mathbb{G}} h^p(x)dx\right)^{\frac{1}{p}}, \tag{4.117}$$

where C is a positive constant independent of f and h.

Proof. Similarly to Theorem 4.15, we need to show

$$\int_{\mathbb{G}} \left(\int_{\mathbb{G}} |x|^{\alpha} |y^{-1}x|^{\lambda} u(y)dy\right)^q dx \leq C \left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(x)dx\right)^{\frac{q}{p}},$$

where $u(y) = h(y)|y|^{\beta}$. We have that

$$\int_{\mathbb{G}} |x|^{\alpha} |y^{-1}x|^{\lambda} u(y)dy \geq \int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\alpha} |y^{-1}x|^{\lambda} u(y)dy,$$

then

$$\left(\int_{\mathbb{G}} |x|^{\alpha} |y^{-1}x|^{\lambda} u(y)dy\right)^q \overset{q<0}{\leq} \left(\int_{B\left(0,\frac{|x|}{2}\right)} |x|^{\alpha} |y^{-1}x|^{\lambda} u(y)dy\right)^q.$$

Hence, we obtain

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} := I_1^{\frac{1}{q}}. \quad (4.118)$$

Similarly with (4.118), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \overset{q<0}{\geq} \left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} := I_2^{\frac{1}{q}}. \quad (4.119)$$

By using (4.118)-(4.119), we get

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} \geq \frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}. \quad (4.120)$$

From now on, in view of Proposition 2.7 we can assume that our quasi-norm is actually a norm.

Step 1. Let us consider I_1. By using Proposition 2.7 with $|y| \leq \frac{|x|}{2}$, we get

$$|y^{-1}x| \overset{(2.10)}{\leq} C(|x|+|y|) \leq \frac{3C}{2}|x| = C_1|x|, \quad (4.121)$$

where $C>0$ and $C_1=\frac{3C}{2}$. Then for any $\lambda<0$, we have

$$C_1^{\lambda}|x|^{\lambda} \leq |y^{-1}x|^{\lambda}.$$

Therefore, we get

$$I_1=\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx \leq C_1^{\lambda q}\int_{\mathbb{G}}|x|^{(\alpha+\lambda)q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}dx.$$

If condition (4.36) in Theorem 4.6 with $u(x)=|x|^{(\alpha+\lambda)q}$ and $v(y)=|y|^{-\beta p}$ in (4.35) is satisfied, then we have

$$I_1 \leq C_1^{\lambda q}\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}u(y)dy\right)^{q}|x|^{(\alpha+\lambda)q}dx \leq C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{q}{p}}.$$

Let us verify condition (4.36). By using assumption $\beta>-\frac{Q}{p'}$, we obtain

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q} > \frac{1}{q}+\frac{\alpha+\lambda}{Q},$$

that is, $\frac{Q+(\alpha+\lambda)q}{Qq} < 0$, then $Q + (\alpha + \lambda)q > 0$ and by using the polar decomposition (2.11):

$$
\begin{aligned}
\left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^{\frac{1}{q}} &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{(\alpha+\lambda)q}dy\right)^{\frac{1}{q}} \\
&= \left(\int_0^{\frac{|x|}{2}} \int_{\mathfrak{S}} r^{Q-1} r^{(\alpha+\lambda)q} dr d\sigma\right)^{\frac{1}{q}} \\
&= \left(|\mathfrak{S}| \int_0^{\frac{|x|}{2}} r^{Q-1+(\alpha+\lambda)q} dr\right)^{\frac{1}{q}} \\
&= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)} |x|^{Q+(\alpha+\lambda)q}\right)^{\frac{1}{q}} \\
&= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} |x|^{\frac{Q+(\alpha+\lambda)q}{q}}.
\end{aligned}
$$

Since $\beta > -\frac{Q}{p'}$, we have

$$
-\beta p(1-p') + Q = \beta p' + Q > 0.
$$

So, $-\beta p(1-p') + Q > 0$. Then, let us consider

$$
\begin{aligned}
\left(\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{-\beta p(1-p')}dy\right)^{\frac{1}{p'}} \\
&= \left(\int_{B\left(0,\frac{|x|}{2}\right)} |y|^{\beta p'}dy\right)^{\frac{1}{p'}} \\
&= \left(\int_0^{\frac{|x|}{2}} \int_{\mathfrak{S}} r^{-\beta p(1-p')} r^{Q-1} dr d\sigma\right)^{\frac{1}{p'}} \\
&= \left(|\mathfrak{S}| \int_0^{\frac{|x|}{2}} r^{\beta p'+Q-1} dr\right)^{\frac{1}{p'}} \\
&= \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)} |x|^{\beta p'+Q}\right)^{\frac{1}{p'}} \\
&= \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}} |x|^{\frac{\beta p'+Q}{p'}}.
\end{aligned}
\tag{4.122}
$$

Therefore by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, we have

$$\begin{aligned}\mathcal{D}_1(|x|) &= \left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B\left(0,\frac{|x|}{2}\right)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}} |x|^{\frac{(\alpha+\lambda)q+Q}{q}} |x|^{\frac{\beta p'+Q}{p'}} \\ &= \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.123}$$

it means $\mathcal{D}_1(|x|)$ is a non-decreasing function. Therefore,

$$D_1 = \inf_{x\neq a} \mathcal{D}_1(|x|) = \left(\frac{|\mathfrak{S}|}{2^{(\alpha+\lambda)q}(Q+(\alpha+\lambda)q)}\right)^{\frac{1}{q}} \left(\frac{|\mathfrak{S}|}{2^{\beta p'+Q}(\beta p'+Q)}\right)^{\frac{1}{p'}} > 0. \tag{4.124}$$

Then by using (4.35), we obtain

$$I_1 \leq 2^{-\lambda q} \int_{\mathbb{G}} |x|^{(\alpha+\lambda)q} \left(\int_{B\left(0,\frac{|x|}{2}\right)} u(y)dy\right)^q dx \leq C\left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(y)dy\right)^{\frac{q}{p}}, \tag{4.125}$$

so that

$$I_1^{\frac{1}{q}} \geq C\left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(y)dy\right)^{\frac{1}{p}} = C\left(\int_{\mathbb{G}} h^p(y)dy\right)^{\frac{1}{p}}. \tag{4.126}$$

Step 2. As in the previous case I_1, now we consider I_2. From $2|x| \leq |y|$, we calculate

$$|y^{-1}x| \overset{(2.10)}{\leq} C(|x|+|y|) \leq \frac{3C}{2}|y| = C_1|y|,$$

then ,

$$|y^{-1}x|^\lambda \geq C|y|^\lambda,$$

where $C>0$. Then, if condition (4.54) with $u(x)=|x|^{\alpha q}$ and $v(y)=|y|^{-(\beta+\lambda)p}$ is satisfied, then we have

$$\begin{aligned}I_2 &= \int_{\mathbb{G}} \left(\int_{\mathbb{G}\setminus B(0,2|x|)} |x|^\alpha |y^{-1}x|^\lambda u(y)dy\right)^q dx \\ &\leq C\int_{\mathbb{G}} |x|^{\alpha q} \left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)|y|^\lambda dy\right)^q dx \leq C\left(\int_{\mathbb{G}} |y|^{-\beta p} u^p(y)dy\right)^{\frac{q}{p}}.\end{aligned}$$

Now let us check condition (4.54). We have

$$\begin{aligned}\left(\int_{\mathbb{G}\setminus B(0,2|x|)} u(y)dy\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G}\setminus B(0,2|x|)} |y|^{\alpha q}dy\right)^{\frac{1}{q}} \\ &= \left(\int_{2|x|}^{\infty}\int_{\mathfrak{S}} r^{\alpha q} r^{Q-1} dr d\sigma\right)^{\frac{1}{q}} \\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}} |x|^{\frac{Q+\alpha q}{q}},\end{aligned}$$

where $Q+\alpha q<0$. From $\alpha>-\frac{Q}{q}$, we have

$$0=\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}>\frac{1}{p'}+\frac{\beta+\lambda}{Q},$$

then

$$(\beta+\lambda)p'+Q<0. \tag{4.127}$$

By using this fact, we have

$$\begin{aligned}\left(\int_{\mathbb{G}\backslash B(0,2|x|)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} &= \left(\int_{\mathbb{G}\backslash B(0,2|x|)} |y|^{-(\beta+\lambda)(1-p')p}dy\right)^{\frac{1}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}.\end{aligned}$$

Then by using $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, we get

$$\begin{aligned}\mathcal{D}_2(|x|) &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}|x|^{\frac{Q+\alpha q}{q}}|x|^{\frac{Q+(\beta+\lambda)p'}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}},\end{aligned} \tag{4.128}$$

it means $\mathcal{D}_2(|x|)$ is a non-increasing function. Therefore we have

$$\begin{aligned}D_2 = \inf_{x\neq a}\mathcal{D}_2(|x|) &= \inf_{x\neq a}\left(\int_{\mathbb{G}\backslash B(0,2|x|)} u(y)dy\right)^{\frac{1}{q}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}}\\ &= \left(\frac{2|\mathfrak{S}|}{|Q+\alpha q|}\right)^{\frac{1}{q}}\left(\frac{2|\mathfrak{S}|}{|Q+(\beta+\lambda)p'|}\right)^{\frac{1}{p'}}>0.\end{aligned} \tag{4.129}$$

Therefore, we have

$$I_2=\int_{\mathbb{G}}\left(\int_{\mathbb{G}\backslash B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx \le 2^{-\lambda q}C_2^q\left(\int_{\mathbb{G}}|y|^{-\beta p}u^p(y)dy\right)^{\frac{q}{p}}.$$

Then, we have

$$I_2^{\frac{1}{q}}\ge 2^{-\lambda}C_2\left(\int_{\mathbb{G}}|y|^{-\beta p}u^p(y)dy\right)^{\frac{1}{p}}=2^{-\lambda}C_2\left(\int_{\mathbb{G}}h^p(y)dy\right)^{\frac{1}{p}}. \tag{4.130}$$

Finally, by using (4.126) and (4.130) in (4.120), we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}|y|^{\beta}h(y)dy\right)^{q}dx\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\\ &\geq \frac{I_1^{\frac{1}{q}}}{2}+\frac{I_2^{\frac{1}{q}}}{2}\\ &\geq C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}\\ &= C\left(\int_{\mathbb{G}}h^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned} \tag{4.131}$$

□

Remark 4.22. *Inequality (4.117) is an even new in the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^N,+)$, $Q=N$ and $|\cdot|=|\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance).*

Remark 4.23. *Particularly, from (4.117) we can not obtain the reverse Hardy-Littlewood-Sobolev inequality from $\alpha>-\frac{Q}{q}>0$.*

4.10. **Improved reverse Stein-Weiss type inequality with $-\infty<q\leq p<0$.** Let us present the improved reverse Stein-Weiss type inequality on homogeneous Lie groups.

Theorem 4.24. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q\geq 1$ and let $|\cdot|$ be an arbitrary homogeneous quasi-norm on $\mathbb{G}$. Assume that $q\leq p<0$, $\lambda<0$, and $\frac{1}{p'}+\frac{1}{q}+\frac{\alpha+\beta+\lambda}{Q}=0$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Then for all non-negative functions $f\in L^{q'}(\mathbb{G})$ and $0<\int_{\mathbb{G}}h^p(x)dx<\infty$, (4.117) holds, that is,*

$$\int_{\mathbb{G}}\int_{\mathbb{G}}|x|^{\alpha}f(x)|y^{-1}x|^{\lambda}h(y)|y|^{\beta}dxdy\geq C\left(\int_{\mathbb{G}}f^{q'}(x)dx\right)^{\frac{1}{q'}}\left(\int_{\mathbb{G}}h^{p}(x)dx\right)^{\frac{1}{p}}, \tag{4.132}$$

if one of the following conditions is satisfied:

(a) $\beta>-\frac{Q}{p'}$;
(b) $\alpha>-\frac{Q}{q}$.

Proof. Let us prove (a). By using (4.130), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}},$$

where $u(y)=|y|^{\beta}h(y)$. Then by using Step 2 in the proof of Theorem 4.21, we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}} &\geq\left(\int_{\mathbb{G}}\left(\int_{\mathbb{G}\setminus B(0,2|x|)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\right)^{\frac{1}{q}}\\ &\overset{\text{Step 2}}{\geq} C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned}$$

Let us prove (b). By using (4.126), we have

$$\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}\geq\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}},$$

where $u(y)=|y|^{\beta}h(y)$. Then by using Step 1 in the proof of Theorem 4.21, we obtain

$$\begin{aligned}\left(\int_{\mathbb{G}}|x|^{\alpha q}\left(\int_{\mathbb{G}}|y^{-1}x|^{\lambda}u(y)dy\right)^{q}dx\right)^{\frac{1}{q}}&\geq\left(\int_{\mathbb{G}}\left(\int_{B\left(0,\frac{|x|}{2}\right)}|x|^{\alpha}u(y)|y^{-1}x|^{\lambda}dy\right)^{q}dx\right)^{\frac{1}{q}}\\&\overset{\text{Step 1}}{\geq}C\left(\int_{\mathbb{G}}|y|^{-\beta p}u^{p}(y)dy\right)^{\frac{1}{p}}.\end{aligned}$$

□

Remark 4.25. *Inequality (4.132) is an even new in the Abelian (Euclidean) case $\mathbb{G}=(\mathbb{R}^N,+)$, $Q=N$ and $|\cdot|=|\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance) with the conditions in Theorem 4.24.*

4.11. **Reverse Hardy inequality with radial derivative on the homogeneous Lie groups.** Let us give reverse Hardy, L^p-Sobolev and L^p-Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{G}$. Suppose that f is a radially decreasing function, i.e., $\mathcal{R}f:=\frac{d}{d|x|}f<0$. Let us give the reverse Hardy inequality on homogeneous Lie groups.

Theorem 4.26 (Reverse Hardy inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q\geq 1$. Assume that $p\in(0,1)$. Then for any non-negative, real-valued and radially decreasing function $f\in C_0^{\infty}(\mathbb{G}\setminus\{0\})$, we have*

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})}\geq\frac{p}{Q-p}\|\mathcal{R}f\|_{L^p(\mathbb{G})}. \tag{4.133}$$

Proof. By denoting $\mathcal{R}_1=-\mathcal{R}$, so that we have $\mathcal{R}_1 f>0$. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{G}}\frac{f^p(x)}{|x|^p}dx&=\int_0^{\infty}\int_{\mathfrak{S}}\frac{f^p(ry)}{r^p}r^{Q-1}drd\sigma(y)\\&=-\frac{p}{Q-p}\int_{\mathbb{G}}\frac{f^{p-1}(x)}{|x|^{p-1}}\mathcal{R}f(x)dx\\&=\frac{p}{Q-p}\int_{\mathbb{G}}\frac{f^{p-1}(x)}{|x|^{p-1}}\mathcal{R}_1 f(x)dx\\&\geq\frac{p}{Q-p}\left\|\frac{f}{|x|}\right\|^{p-1}_{L^p(\mathbb{G})}\|\mathcal{R}_1 f\|_{L^p(\mathbb{G})}.\end{aligned} \tag{4.134}$$

This gives

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})}\geq\frac{p}{Q-p}\|\mathcal{R}_1 f\|_{L^p(\mathbb{G})}, \tag{4.135}$$

implying (4.133). □

4.12. **Reverse L^p-Sobolev inequality with radial derivative on the homogeneous Lie groups.** Let us define by $\mathbb{E} = |x|\mathcal{R}$ the Euler operator. Then we have the reverse L^p-Sobolev inequality.

Theorem 4.27 (Reverse L^p-Sobolev inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0,1)$. Then for any nonnegative, real-valued and radially decreasing function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have*

$$\|f\|_{L^p(\mathbb{G})} \geq \frac{p}{Q}\|\mathbb{E}f\|_{L^p(\mathbb{G})}. \tag{4.136}$$

Proof. By denote $\mathbb{E}_1 = |x|\mathcal{R}_1$, so that $\mathbb{E}_1 f > 0$. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}
\int_{\mathbb{G}} f^p(x)dx &= \int_0^\infty \int_{\mathfrak{S}} f^p(ry)r^{Q-1}drd\sigma(y) \\
&= -\frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)|x|\mathcal{R}f(x)dx \\
&= \frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)|x|\mathcal{R}_1 f(x)dx \\
&= \frac{p}{Q}\int_{\mathbb{G}} f^{p-1}(x)\mathbb{E}_1 f(x)dx \\
&\geq \frac{p}{Q}\,\|f\|^{p-1}_{L^p(\mathbb{G})}\,\|\mathbb{E}_1 f\|_{L^p(\mathbb{G})}.
\end{aligned} \tag{4.137}$$

This gives

$$\|f\|_{L^p(\mathbb{G})} \geq \frac{p}{Q}\|\mathbb{E}_1 f\|_{L^p(\mathbb{G})}, \tag{4.138}$$

implying (4.136). ∎

4.13. **Reverse L^p-Caffarelli-Kohn-Nirenberg inequality on the homogeneous Lie groups.** Let us give the reverse L^p-Caffarelli-Kohn-Nirenberg inequality on $\mathbb{G}$.

Theorem 4.28 (Reverse L^p-Caffarelli-Kohn-Nirenberg inequality)**.** *Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0,1)$. Then for any nonnegative, real-valued and radially decreasing function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have*

$$\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^p_{L^p(\mathbb{G})} \geq \frac{p}{Q-\gamma}\left\|\frac{\mathcal{R}f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{4.139}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\gamma = \alpha + \beta + 1$, such that $Q > \gamma$.

Proof. By combining polar decomposition (2.11), integration by parts and reverse Hölder's inequality, we get

$$\begin{aligned}
\int_{\mathbb{G}} \frac{f^p(x)}{|x|^\gamma}dx &= \int_0^\infty \int_{\mathfrak{S}} \frac{f^p(ry)}{r^\gamma} r^{Q-1} dr d\sigma(y) \\
&= -\frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\gamma-1}}\mathcal{R}f(x)dx \\
&= \frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\alpha+\beta}}\mathcal{R}_1 f(x)dx \\
&= \frac{p}{Q-\gamma}\int_{\mathbb{G}} \frac{f^{p-1}(x)}{|x|^{\beta}}\frac{\mathcal{R}_1 f(x)}{|x|^\alpha}dx \\
&\geq \frac{p}{Q-\gamma}\left\|\frac{f}{|x|^{\frac{\beta p'}{p}}}\right\|^{p-1}_{L^p(\mathbb{G})}\left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})} \\
&= \frac{p}{Q-\gamma}\left\|\frac{f}{|x|^{\frac{\beta\frac{p}{p-1}}{p}}}\right\|^{p-1}_{L^p(\mathbb{G})}\left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})} \\
&= \frac{p}{Q-\gamma}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}\left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}.
\end{aligned} \tag{4.140}$$

This gives

$$\left\|\frac{f}{|x|^{\frac{\gamma}{p}}}\right\|^{p}_{L^p(\mathbb{G})} \geq \frac{p}{Q-\gamma}\left\|\frac{\mathcal{R}_1 f}{|x|^\alpha}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x|^{\frac{\beta}{p-1}}}\right\|^{p-1}_{L^p(\mathbb{G})}, \tag{4.141}$$

which implies (4.139). □

Remark 4.29. *In (4.139), if we take $\gamma = p$ and $\alpha = 0$, then we have the reverse Hardy inequality. Also, if we take $\gamma = 0$ and $\beta = 0$, then we have the reverse L^p-Sobolev inequality.*

5. Applications

In this chapter, we show some applications of the fractional functional inequalities in PDE.

5.1. **Lyapunov-type inequality for the fractional p-sub-Laplacian.** In the one of the popular Lyapunov's work [67], he considered the following one-dimensional homogeneous Dirichlet boundary value problem was studied (for the second order ODE)

$$\begin{cases} u''(x) + \omega(x)u(x) = 0, \;\; x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{5.1}$$

and it was proved that, if u is a non-trivial solution of (5.1) and $\omega(x)$ is a real-valued and continuous function on $[a,b]$, then

$$\int_a^b |\omega(x)|dx > \frac{4}{b-a}. \tag{5.2}$$

Inequality (5.2) is called a (classical) Lyapunov inequality. This inequality has an application in spectral theory. If $\omega(x) = \lambda$, where λ is a positive constant, then we get lower estimate for the first eigenvalue of the problem (5.1) in the following form:

$$\lambda_1 > \frac{4}{(b-a)^2}.$$

Now, Lyapunov's inequality has a lot of extensions in one-dimensional and multi-dimensional cases. As example, in the work [68] the author obtains the Lyapunov inequality for the one-dimensional Dirichlet p-Laplacian

$$\begin{cases} (|u'(x)|^{p-2}u'(x))' + \omega(x)u^{p-1}(x) = 0, \;\; x \in (a,b), \;\; 1 < p < \infty, \\ u(a) = u(b) = 0, \end{cases} \tag{5.3}$$

where $\omega(x) \in L^1(a,b)$, so

$$\int_a^b |\omega(x)|dx > \frac{2^p}{(b-a)^{p-1}}, \;\; 1 < p < \infty. \tag{5.4}$$

Particularly, if $p = 2$ in (5.4), we recover (5.2).

In the paper [69] the authors obtained interesting results concerning Lyapunov inequalities for the multi-dimensional fractional p-Laplacian $(-\Delta_p)^s$, $1 < p < \infty$, $s \in (0,1)$, with a homogeneous Dirichlet boundary condition, that is,

$$\begin{cases} (-\Delta_p)^s u = \omega(x)|u|^{p-2}u, \;\; x \in \Omega, \\ u(x) = 0, \;\; x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.5}$$

where $\Omega \subset \mathbb{R}^N$ is a measurable set, $1 < p < \infty$, and $s \in (0,1)$. Let us recall the following result of [69].

Theorem 5.1. *Let $\omega \in L^{\theta}(\Omega)$ with $N > sp$, $\frac{N}{sp} < \theta < \infty$, be a non-negative weight. Suppose that problem (5.5) has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then*

$$\left(\int_{\Omega} \omega^{\theta}(x)\, dx\right)^{\frac{1}{\theta}} > \frac{C}{r_{\Omega}^{sp-\frac{N}{\theta}}}, \tag{5.6}$$

where $C > 0$ is a universal constant and r_{Ω} is the inner radius of Ω.

In this section we prove a Lyapunov-type inequality for the fractional p-sub-Laplacian with a homogeneous Dirichlet boundary problem on $\mathbb{G}$. Assume $p > 1$ and $s \in (0, 1)$ be such that $Q > sp$ and $\Omega \subset \mathbb{G}$ be a Haar measurable set. We denote by $r_{\Omega,q}$ the inner quasi-radius of Ω, that is,

$$r_{\Omega,q} = \max\{|x| : \ x \in \Omega\}. \tag{5.7}$$

Let us consider

$$\begin{cases} (-\Delta_p)^s u(x) = \omega |u(x)|^{p-2} u(x), \ x \in \Omega, \\ u(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \end{cases} \tag{5.8}$$

where $\omega \in L^{\infty}(\Omega)$.

Definition 5.2. A function $u \in W_0^{s,p}(\Omega)$ is called a weak solution of the problem (5.8) if

$$\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|y^{-1}x|^{Q+sp}} dxdy = \int_{\Omega} \omega(x)|u(x)|^{p-2}u(x)v(x)dx \tag{5.9}$$

for all $v \in W_0^{s,p}(\Omega)$.

Then we have the following theorem:

Theorem 5.3. *Let $\Omega \subset \mathbb{G}$ be a Haar measurable set. Let $\omega \in L^{\theta}(\Omega)$ be a non-negative weight with $\frac{Q}{sp} < \theta < \infty$. Suppose that problem (5.8) with $Q > ps$ has a non-trivial weak solution $u \in W_0^{s,p}(\Omega)$. Then, we have*

$$\|\omega\|_{L^{\theta}(\Omega)} \geq \frac{C}{r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.10}$$

where $C = C(Q, p, s) > 0$.

Proof. By denoting

$$\beta = \alpha p + (1-\alpha)p^*,$$

where $\alpha = \frac{\theta-\theta/sp}{\theta-1} \in (0, 1)$ and p^* is the Sobolev conjugate exponent as in Theorem 3.9. Assume that $\beta = p\theta'$ with $1/\theta + 1/\theta' = 1$. Then, we have

$$\int_{\Omega} \frac{|u(x)|^{\beta}}{r_{\Omega,q}^{\alpha sp}} dx \leq \int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx. \tag{5.11}$$

By using Hölder's inequality with exponents $\nu = \alpha^{-1}$ and $1/\nu + 1/\nu' = 1$, we get

$$\int_{\Omega} \frac{|u(x)|^{\beta}}{|x|^{\alpha sp}} dx \leq \int_{\Omega} \frac{|u(x)|^{\alpha p}|u(x)|^{(1-\alpha)p^*}}{|x|^{\alpha sp}} dx \leq \left(\int_{\Omega} \frac{|u(x)|^{p}}{|x|^{sp} dx}\right)^{\alpha} \left(\int_{\Omega} |u(x)|^{p^*} dx\right)^{1-\alpha}. \tag{5.12}$$

Then, by combining Theorem 3.9 and 3.4, we have

$$\begin{aligned}
\int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx &\leq C_1^\alpha \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|y^{-1}x|^{Q+sp}}dxdy\right)^{\alpha/p} C_2^{(1-\alpha)p^*/p}[u]_{s,p}^{(1-\alpha)p^*/p}\\
&\leq C_1^\alpha [u]_{s,p}^\alpha C_2^{(1-\alpha)p^*/p}[u]_{s,p}^{(1-\alpha)p^*/p}\\
&= C\left([u]_{s,p}^p\right)^{(\alpha p+(1-\alpha)p^*)/p}\\
&= C\left(\int_\Omega \omega(x)|u(x)|^p dx\right)^{\theta'}\\
&\leq C\left(\int_\Omega \omega^\theta(x)dx\right)^{\theta'/\theta}\int_\Omega |u(x)|^{p\theta'}dx\\
&= C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx.
\end{aligned}$$

It means, we have

$$\int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx \leq C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx.$$

Thus, from (5.11) we get

$$\frac{1}{r_{\Omega,q}^{\alpha sp}}\int_\Omega |u(x)|^\beta dx \leq \int_\Omega \frac{|u(x)|^\beta}{|x|^{\alpha sp}}dx \leq C\|\omega\|_{L^\theta(\Omega)}^{\theta'}\int_\Omega |u(x)|^\beta dx. \tag{5.13}$$

Finally, we arrive at

$$\frac{C}{r_{\Omega,q}^{sp-Q/\theta}} \leq \|\omega\|_{L^\theta(\Omega)}. \tag{5.14}$$

Theorem 5.3 is proved. $\square$

Let consider the following spectral problem for the non-linear, fractional p-sub-Laplacian $(-\Delta_p)^s$, $1<p<\infty$, $s\in(0,1)$, with Dirichlet boundary condition:

$$\begin{cases}(-\Delta_p)^s u = \lambda|u|^{p-2}u, \ x\in\Omega,\\ u(x)=0, \ x\in\mathbb{G}\setminus\Omega.\end{cases} \tag{5.15}$$

We have the following Rayleigh quotient for the fractional Dirichlet p-sub-Laplacian (cf. [69])

$$\lambda_1 = \inf_{u\in W_0^{s,p}(\Omega),\, u\neq 0}\frac{[u]_{s,p}^p}{\|u\|_{L^p(\mathbb{G})}^p}. \tag{5.16}$$

As a consequence of Theorem 5.3 we obtain the following theorem:

Theorem 5.4. *Assume λ_1 be the first eigenvalue of problem (5.15) given by (5.16). Assume $Q>sp$, $s\in(0,1)$ and $1<p<\infty$. Then we have*

$$\lambda_1 \geq \sup_{\frac{Q}{sp}<\theta<\infty}\frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.17}$$

where C is a positive constant given in Theorem 5.3, $|\cdot|$ is the Haar measure and $r_{\Omega,q}$ is the inner quasi-radius of Ω.

Proof. In Theorem 5.3, by taking $\omega = \lambda_1 \in L^{\theta}(\Omega)$ and using Lyapunov-type inequality (5.10), we get that

$$\|\omega\|_{L^{\theta}(\Omega)} = \|\lambda_1\|_{L^{\theta}(\Omega)} = \left(\int_{\Omega} \lambda_1^{\theta} dx\right)^{1/\theta} \geq \frac{C}{r_{\Omega,q}^{sp-Q/\theta}}. \tag{5.18}$$

For every $\theta > \frac{Q}{sp}$, we have

$$\lambda_1 \geq \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,q}^{sp-Q/\theta}}. \tag{5.19}$$

Thus, we get

$$\lambda_1 \geq \sup_{\frac{Q}{sp}<\theta<\infty} \frac{C}{|\Omega|^{\frac{1}{\theta}} r_{\Omega,q}^{sp-Q/\theta}}, \tag{5.20}$$

for all $\frac{Q}{sp} < \theta < \infty$.

Theorem 5.4 is proved. ◻

5.2. **Lyapunov-type inequality for the fractional p-sub-Laplacian system.** Historically, in the work [70], at the first time authors showed Lyapunov's inequality for the system. They considered a system of ODE for p and q-Laplacian on the interval (a,b) with the homogeneous Dirichlet condition in the following form:

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x)|u(x)|^{\alpha-2}u(x)|v(x)|^{\beta}, \\ -(|v'(x)|^{q-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), \end{cases} \tag{5.21}$$

on the interval (a,b), with

$$u(a) = u(b) = v(a) = v(b) = 0, \tag{5.22}$$

where $f,g \in L^1(a,b)$, $f,g \geq 0$, $p,q > 1$, α, $\beta \geq 0$ and

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

So, for the system (5.21) with Dirichlet condition (5.22), we have the following estimate (Lyapunov's inequality):

$$2^{\alpha+\beta} \leq (b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}} \left(\int_a^b f(x)dx\right)^{\frac{\alpha}{p}} \left(\int_a^b g(x)dx\right)^{\frac{\beta}{q}}, \tag{5.23}$$

where $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. For the more general Lyapunov's inequality for fractional p-Laplacian with homogeneous Dirichlet conditions was proved in [71]. In the previous section, we proved a Lyapunov-type inequality for the fractional p-sub-Laplacian with the homogeneous Dirichlet condition. Here we establish Lyapunov-type inequality for the fractional p-sub-Laplacian system for the homogeneous Dirichlet problem.

Namely, let us consider the fractional p-sub-Laplacian system:

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = \omega_1(x)|u_1(x)|^{\alpha_1-2}u_1(x)|u_2(x)|^{\alpha_2}\dots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ (-\Delta_{p_2})^{s_2} u_2(x) = \omega_2(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2-2}u_2(x)\dots|u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ \qquad \dots \\ (-\Delta_{p_n})^{s_n} u_n(x) = \omega_n(x)|u_1(x)|^{\alpha_1}|u_2(x)|^{\alpha_2}\dots|u_n(x)|^{\alpha_n-2}u_n(x), \ x \in \Omega, \end{cases} \tag{5.24}$$

with homogeneous Dirichlet conditions

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \quad i = 1, \dots, n, \tag{5.25}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\omega_i \in L^1(\Omega)$, $\omega_i \geq 0$, $s_i \in (0,1)$, $p_i \in (1,\infty)$ and $(-\Delta_p)^s$ is the fractional p-sub-Laplacian on $\mathbb{G}$. Here $B(x,\delta)$ is a quasi-ball with respect to q, with radius δ, centred at $x \in \mathbb{G}$, and α_i are positive parameters such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{p_i} = 1. \tag{5.26}$$

We denote by $r_{\Omega,q}$ the inner quasi-radius of Ω.

Definition 5.5. We say that $(u_1, \dots, u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$ is a weak solution of (5.24)-(5.25) if for all $(v_1, \dots, v_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$, we have

$$\begin{aligned} &\int_{\mathbb{G}} \int_{\mathbb{G}} \frac{|u_i(x) - u_i(y)|^{p_i - 2}(u_i(x) - u_i(y))(v_i(x) - v_i(y))}{|y^{-1}x|^{Q+s_i p_i}} dxdy \\ &= \int_{\Omega} \omega_i(x) \left(\prod_{j=1}^{i-1} |u_j(x)|^{\alpha_j} \right) \left(\prod_{j=i+1}^{n} |u_j(x)|^{\alpha_j} \right) |u_i(x)|^{\alpha_i - 2} u_i(x) v_i(x) dx, \end{aligned} \tag{5.27}$$

for every $i = 1, \dots, n$.

Now we present the following analogue of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$.

Theorem 5.6. *Let $s_i \in (0,1)$ and $p_i \in (1,\infty)$ be such that $Q > s_i p_i$ for all $i = 1, \dots, n$. Let $\omega_i \in L^\theta(\Omega)$ be a non-negative weight and assume that*

$$1 < \max_{i=1,\dots,n} \left\{ \frac{Q}{s_i p_i} \right\} < \theta < \infty.$$

If (5.24)-(5.25) admits a nontrivial weak solution, then

$$\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta \alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q - \theta \sum_{j=1}^n s_j \alpha_j}, \tag{5.28}$$

where $C > 0$ is a positive constant.

Remark 5.7. *In Theorem 5.6, by taking $n = 1$ and $\alpha_1 = p$, we establish the Lyapunov-type inequality for the fractional p-sub-Laplacian on $\mathbb{G}$ (see, e.g. Theorem 5.3).*

Proof of Theorem 5.6. For all $i = 1, \dots, n$, let us denote

$$\xi_i = \gamma_i p_i + (1 - \gamma_i) p_i^*, \tag{5.29}$$

and

$$\gamma_i = \frac{\theta - \frac{Q}{s_i p_i}}{\theta - 1}, \tag{5.30}$$

where $p_i^* = \frac{Q}{Q-s_ip_i}$ is the Sobolev conjugate exponent as in Theorem 3.9. For all $i = 1, \ldots, n$ we have $\gamma_i \in (0,1)$ and $\xi_i = p_i\theta'$, where $\theta' = \frac{\theta}{\theta-1}$. Then for every $i \in \{1, \ldots, n\}$ we have

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}} dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx,$$

and from Hölder's inequality with the following exponents $\nu_i = \frac{1}{\gamma_i}$ and $\frac{1}{\nu_i} + \frac{1}{\nu_i'} = 1$, we get

$$\begin{aligned}\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx &= \int_\Omega \frac{|u_i(x)|^{\gamma_i p_i} |u_i(x)|^{(1-\gamma_i)p_i^*}}{|x|^{\gamma_i s_i p_i}} dx \\ &\leq \left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}} dx \right)^{\gamma_i} \left(\int_\Omega |u_i(x)|^{p_i^*} dx \right)^{1-\gamma_i}. \quad (5.31)\end{aligned}$$

On the other hand, from Theorem 3.9, we obtain

$$\left(\int_\Omega |u_i(x)|^{p_i^*} dx \right)^{1-\gamma_i} \leq C[u_i]_{s_i,p_i}^{p_i^*(1-\gamma_i)},$$

and from Theorem 3.4, we have

$$\left(\int_\Omega \frac{|u_i(x)|^{p_i}}{|x|^{s_i p_i}} dx \right)^{\gamma_i} \leq C[u_i]_{s_i,p_i}^{p_i \gamma_i}.$$

Thus, from (5.31) and by taking $u_i(x) = v_i(x)$ in (5.27), we get

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} \leq C([u_i]_{s_i,p_i,\Omega}^{p_i})^{\frac{\xi_i}{p_i}} \leq C([u_i]_{s_i,p_i}^{p_i})^{\frac{\xi_i}{p_i}}$$

$$= C \left(\int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} dx \right)^{\frac{\xi_i}{p_i}} = C \left(\int_\Omega \omega_i(x) \prod_{j=1}^n |u_j|^{\alpha_j} dx \right)^{\theta'},$$

for every $i = 1, \ldots, n$. Hence, by using Hölder's inequality with exponents θ and θ', we obtain

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j \theta'} dx.$$

By using Hölder's inequality and (5.26), we get

$$\int_\Omega \prod_{j=1}^n |u_j(x)|^{\alpha_j \theta'} dx \leq \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx \right)^{\frac{\alpha_j}{p_j}}.$$

This implies that

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx \leq C \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \prod_{j=1}^n \left(\int_\Omega |u_j|^{\theta' p_j} dx \right)^{\frac{\alpha_j}{p_j}}.$$

So we establish

$$\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}} dx \leq \int_\Omega \frac{|u_i(x)|^{\xi_i}}{|x|^{\gamma_i s_i p_i}} dx$$

$$\leq C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta}{\theta-1}} \prod_{j=1}^{n} \left(\int_\Omega |u_j|^{\theta' p_j} dx \right)^{\frac{\alpha_j}{p_j}}.$$

Thus, for every $e_i > 0$ we have

$$\left(\int_\Omega \frac{|u_i(x)|^{\xi_i}}{r_{\Omega,q}^{\gamma_i s_i p_i}} dx \right)^{e_i} = \frac{1}{r_{\Omega,q}^{e_i \gamma_i s_i p_i}} \left(\int_\Omega |u_i(x)|^{\xi_i} dx \right)^{e_i}$$

$$\leq C\|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \prod_{j=1}^{n} \left(\int_\Omega |u_j|^{\theta' p_j} dx \right)^{\frac{e_i\alpha_j}{p_j}},$$

so that

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \prod_{i=1}^{n} \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx \right)^{e_i}$$

$$\leq C \left(\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \right) \left(\prod_{i=1}^{n} \left(\int_\Omega |u_i(x)|^{\theta' p_i} dx \right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i}} \right).$$

This yields

$$\frac{1}{r_{\Omega,q}^{\sum_{j=1}^n \gamma_j s_j p_j e_j}} \leq C \left(\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{e_i\theta}{\theta-1}} \right) \left(\prod_{i=1}^{n} \left(|u_i(x)|^{\theta' p_i} dx \right)^{\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i} \right), \tag{5.32}$$

where C is a positive constant. Then, let us choose e_i, $i = 1, \ldots, n$, such that

$$\frac{\alpha_i \sum_{j=1}^n e_j}{p_i} - e_i = 0, \quad i = 1, \ldots, n.$$

Consequently, by using (5.26) we have the solution of this system

$$e_i = \frac{\alpha_i}{p_i}, \quad i = 1, \ldots, n. \tag{5.33}$$

By combining (5.32), (5.30) and (5.33) we establish

$$\prod_{i=1}^{n} \|\omega_i\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta \sum_{j=1}^n s_j \alpha_j}. \tag{5.34}$$

Theorem 5.3 is proved. □

Now, let us discuss an application of the Lyapunov-type inequality for the fractional p-sub-Laplacian system on $\mathbb{G}$. In order to do it we consider the spectral problem for the fractional p-sub-Laplacian system in the following form:

$$\begin{cases} (-\Delta_{p_1})^{s_1} u_1(x) = \lambda_1 \alpha_1 \varphi(x) |u_1(x)|^{\alpha_1 - 2} u_1(x) |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ (-\Delta_{p_2})^{s_2} u_2(x) = \lambda_2 \alpha_2 \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2 - 2} u_2(x) \ldots |u_n(x)|^{\alpha_n}, \ x \in \Omega, \\ \qquad \qquad \ldots \\ (-\Delta_{p_n})^{s_n} u_n(x) = \lambda_n \alpha_n \varphi(x) |u_1(x)|^{\alpha_1} |u_2(x)|^{\alpha_2} \ldots |u_n(x)|^{\alpha_n - 2} u_n(x), \ x \in \Omega, \end{cases} \tag{5.35}$$

with

$$u_i(x) = 0, \quad x \in \mathbb{G} \setminus \Omega, \ \ i = 1, \ldots, n, \tag{5.36}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set, $\varphi \in L^1(\Omega)$, $\varphi \geq 0$ and $s_i \in (0,1)$, $p_i \in (1,\infty)$, $i = 1,\dots,n$.

Definition 5.8. We say that $\lambda = (\lambda_1,\dots,\lambda_n)$ is an eigenvalue if the problem (5.35)-(5.36) admits at least one nontrivial weak solution $(u_1,\dots,u_n) \in \prod_{i=1}^n W_0^{s_i,p_i}(\Omega)$.

Theorem 5.9. *Let $s_i \in (0,1)$ and $p_i \in (1,\infty)$ be such that $Q > s_i p_i$, for all $i = 1,\dots,n$, and*

$$1 < \max_{i=1,\dots,n}\left\{\frac{Q}{s_i p_i}\right\} < \theta < \infty.$$

Let $\varphi \in L^\theta(\Omega)$ with $\|\varphi\|_{L^\theta(\Omega)} \neq 0$. Then, we have

$$\lambda_k \geq \frac{C}{\alpha_k}\left(\frac{1}{\prod_{i=1,i\neq k}^n \lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_k}{\alpha_k}}\left(\frac{1}{r_{\Omega,q}^{\theta\sum_{i=1}^n \alpha_i s_i - Q}\prod_{i=1,i\neq k}^n \alpha_i^{\frac{\theta\alpha_i}{p_i}}\int_\Omega \varphi^\theta(x)dx}\right)^{\frac{p_k}{\theta\alpha_k}}, \quad (5.37)$$

where C is a positive constant and $k = 1,\dots,n$.

Proof of Theorem 5.9. In Theorem 5.6 by taking $\omega_k = \lambda_k\alpha_k\varphi(x)$, $k = 1,\dots,n$, we have

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}}\lambda_k^{\frac{\theta\alpha_k}{p_k}}\prod_{i=1,i\neq k}^n(\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}}\prod_{i=1}^n\|\varphi\|_{L^\theta(\Omega)}^{\frac{\theta\alpha_i}{p_i}} \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}.$$

Thus, using (5.26) we obtain

$$\alpha_k^{\frac{\theta\alpha_k}{p_k}}\lambda_k^{\frac{\theta\alpha_k}{p_k}}\prod_{i=1,i\neq k}^n(\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}}\int_\Omega\varphi^\theta(x)dx \geq C r_{\Omega,q}^{Q-\theta\sum_{j=1}^n s_j\alpha_j}.$$

This implies

$$\lambda_k^{\frac{\theta\alpha_k}{p_k}} \geq \frac{C}{\alpha_k^{\frac{\theta\alpha_k}{p_k}} r_{\Omega,q}^{\theta\sum_{j=1}^n s_j\alpha_j - Q}\prod_{i=1,i\neq k}^n(\alpha_i\lambda_i)^{\frac{\theta\alpha_i}{p_i}}\int_\Omega\varphi^\theta(x)dx}, \quad k = 1,\dots,n.$$

Finally, we obtain that

$$\lambda_k \geq \frac{C}{\alpha_k}\left(\frac{1}{\prod_{i=1,i\neq k}^n \lambda_i^{\frac{\alpha_i}{p_i}}}\right)^{\frac{p_k}{\alpha_k}}\left(\frac{1}{r_{\Omega,q}^{\theta\sum_{i=1}^n \alpha_i s_i - Q}\prod_{i=1,i\neq k}^n \alpha_i^{\frac{\theta\alpha_i}{p_i}}\int_\Omega \varphi^\theta(x)dx}\right)^{\frac{p_k}{\theta\alpha_k}},$$

$$k = 1,\dots,n. \quad (5.38)$$

Theorem 5.9 is proved. ∎

5.3. **Existence of weak solutions with nonlocal source on the Heisenberg and stratified groups.** In [72], under certain assumptions on f (classically, this condition is called Ambrosetti-Rabinowitz condition), for the following semilinear equation

$$\begin{cases} -\Delta u = f(x,u), \ x \in \Omega \subset \mathbb{R}^n, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \quad (5.39)$$

the authors proved existence of solutions by the mountain pass theorem. Mountain pass theorem is using to show critical points of the some differentiable functional. Here and after by $\partial\Omega$ we denote the boundary of a smooth bounded set Ω. After the work of Ambrosetti and Rabinowitz [72], a number of extensions and generalisations of their result has been published. Also, for the fractional nonlinear problems, for the fractional $p-$Laplacian, fractional Schrödinger–Kirchhoff type and Choquard-Kirchhoff existence of weak solutions were proved in [73], [74], [75] and [76]. One of the main aim of this section is to extend the above ideas to non-commutative analysis, it means using our functional inequalities. Hence, we will consider analogues problems on the Heisenberg group, which is the most popular example of the non-Abelian nilpotent Lie groups. On the Heisenberg group, there is already a number of results related to the existence of solutions to the semilinear equations starting the pioneering works (see e.g., [77] and [78]). In this section we show existence of the weak solution by mountain pass theorem on Heisenberg group, which can be easily extended to the general stratified Lie groups.

Firstly, let us give definition of the Palais-Smale sequence (shortly, $(PS)_c$) sequence.

Definition 5.10. [72] Let E be a Banach space. A sequence $\{u_n\}$ is a $(PS)_c$ sequence for a functional $\Phi \in (\Phi, \mathbb{R})$, if every $\{u_n\} \subset E$ satisfies:

$$\Phi(u_n) \to c, \quad \text{for} \quad n \to \infty, \tag{5.40}$$

and

$$\Phi'(u_n) \to 0, \quad \text{for} \quad n \to \infty \quad \text{in } E^*, \tag{5.41}$$

where $'$ is the Fréchet differential and E^* is the dual space of E.

Then let us give a version of the (minimax) mountain pass theorem (see, e.g. [79]).

Theorem 5.11. *Suppose that X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1-functional with a $(PS)_c$ sequence. Let Γ be a class of paths joining $u = 0$ with $u = \omega$:*

$$\Gamma := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0, \ \gamma(1) = \omega\}, \tag{5.42}$$

where $\omega \in X$, $\|\omega\| > r > 0$, Φ is bounded from below on $S(0, \rho) = \{u \in X : \|u\| \leq \rho\}$, that is,

$$\alpha = \max\{\Phi(0), \Phi(\omega)\} < \inf_{u \in S(0,\rho)} \Phi(u) = \beta. \tag{5.43}$$

Then Φ possesses a critical value $c \geq \beta$ which can be characterised as

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma(0,1)} \Phi(u).$$

5.3.1. *On Heisenberg group.* It is well-known that the class of the Heisenberg groups is a subclass of the stratified Lie groups, that is, obviously, the above theorem is valid for the Heisenberg group setting. Firstly, we show our result on Heisenberg group.

Assume that $f(x, \xi)$ is a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions (Ambrosetti-Rabinowitz condition):

- $p1)$ There exist constants a_1, $a_2 > 0$ such that $|f(x, \xi)| \leq a_1 + a_2|\xi|^s$, a.e. $x \in \Omega$ and $\xi \in \mathbb{R}$ with $p < s < \frac{Qp}{Q-p} - 1$;
- $p2)$ $\lim_{|\xi|\to 0} \frac{f(x,\xi)}{|\xi|^{p-1}} = 0$, uniformly in $x \in \Omega$;

$p3)$ There exist $\mu > p$ and $r > 0$ such that $0 < \mu F(x,\xi) < \xi f(x,\xi)$ with $|\xi| > r$, a.e. $x \in \mathbb{H}^n$, $\xi \in \mathbb{R}$. Here $F(x,\xi) = \int_0^\xi f(x,t)dt$.

$p4)$ $f(x,\xi) \in C(\overline{\Omega}, \mathbb{R})$.

As the model case, the function $f(x,\xi) = a(x)|\xi|^{s-2}\xi$ with $a \in L^\infty(\Omega)$ and $s \in [p, p^*)$ can be considered as a Carathéodory function satisfying the assumptions $p1)$-$p4)$.

Then, under the above assumptions on the right hand side, we consider the following homogeneous Dirichlet boundary value problem for the p-sub-Laplacian with the nonlinear source (or the nonlinear right hand side) on the Heisenberg group:

$$\begin{cases} -\Delta_{H,p} u = f(x,u), \ x \in \Omega \subset \mathbb{H}^n, \ 1 < p < Q, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.44}$$

where $\Delta_{H,p}$ is defined in (2.30). Let us recall from Section 2.3 the Sobolev space in the following form:

$$S^{1,p}(\Omega) := \{u \in L^p(\Omega) : X_i u \in L^p(\Omega) \text{ and } Y_i u \in L^p(\Omega), \ i = 1, \dots, n\} \tag{5.45}$$

with the norm

$$\|u\|_{S^{1,p}(\Omega)} = \left(\int_\Omega |u(x)|^p + |\nabla_H u(x)|^p dx \right)^{\frac{1}{p}}. \tag{5.46}$$

Let $S_0^{1,p}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{S_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_H u(x)|^p dx \right)^{\frac{1}{p}}. \tag{5.47}$$

For simplicity, we also use the notation $W := S_0^{1,p}(\Omega)$.

Note that the above integral measure is indeed the standard Lebesque measure since it can be considered as a Haar measure on $\mathbb{H}^n$, that is, the Lebesque measure is also translation invariant with respect to the group law of $\mathbb{H}^n$.

To introduce a variational structure for problem (5.44), we introduce $I : W \to \mathbb{R}$ as follows

$$I(u) := \frac{1}{p} \int_\Omega |\nabla_H u|^p dx - \int_\Omega F(x,u) dx, \tag{5.48}$$

where

$$F(x,u) = \int_0^u f(x,\xi) d\xi.$$

We note I is a Fréchet differentiable functional with respect to $u \in W$ for any $\varphi \in W$, so we have

$$\langle I'(u), \varphi \rangle = \int_\Omega |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H \varphi dx - \int_\Omega f(x,u)\varphi(x) dx, \tag{5.49}$$

where $\langle \cdot, \cdot \rangle$ is the dual product between W and its dual space W^*. Let us give the definition of a weak solution.

Definition 5.12. We say $u : \Omega \to \mathbb{R}$ is a weak solution of (5.44), if $u \in W$, such that

$$\int_\Omega |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H \varphi dx = \int_\Omega f(x,u)\varphi(x) dx, \ \ \forall \varphi \in C_c^\infty(\Omega). \tag{5.50}$$

Then we have the following properties of Carathéodory functions:

Lemma 5.13. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f is a Carathéodory function and assumption p3) holds true, then there exist constants $a_3, a_4 > 0$ such that*

$$a_3|\xi|^{\mu} - a_4 \leq F(x,\xi), \quad \forall x \in \Omega, \tag{5.51}$$

where $\mu > p$.

Lemma 5.14. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f be a Carathéodory function satisfying assumptions p1) and p4). Then for any $\xi \in \mathbb{R}$, we have*

$$|f(x,\xi)| \leq \varepsilon|\xi|^{p-1} + (s+1)\kappa(\varepsilon)|\xi|^{s}, \tag{5.52}$$

and

$$|F(x,\xi)| \leq \varepsilon|\xi|^{p} + \kappa(\varepsilon)|\xi|^{s+1}, \tag{5.53}$$

where ε and $\kappa(\varepsilon)$ are some positive small numbers. Here the numbers s and p are defined as in p1).

Note that the proofs of Lemma 5.13 and Lemma 5.14 are exactly the same as the Euclidean case in [72].

Let us check the first assumption of the mountain pass theorem.

Lemma 5.15. *Let Ω be a measurable set in $\mathbb{H}^n$. Assume that f be a Carathéodory function satisfying the assumptions p1) and p2). Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$.*

Proof. By using Lemma 5.14 in (5.48), we get

$$\begin{aligned} I(u) = \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - \int_\Omega F(x,u(x))dx \geq \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - \varepsilon\int_\Omega |u(x)|^p dx \\ - \kappa(\varepsilon)\int_\Omega |u(x)|^{s+1}dx. \end{aligned} \tag{5.54}$$

From $1 < p < p^*$ and Ω is a measurable domain, we have the continuous embedding $L^{p^*}(\Omega) \hookrightarrow L^p(\Omega)$ in $\Omega \subset \mathbb{H}^n$. For $s+1 < p^*$ we also have the following continuous embedding $L^{p^*}(\Omega) \hookrightarrow L^{s+1}(\Omega)$, then

$$\begin{aligned} I(u) &= \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - \int_\Omega F(x,u(x))dx \\ &\geq \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - \varepsilon\int_\Omega |u(x)|^p dx - \kappa(\varepsilon)\int_\Omega |u(x)|^{s+1}dx \\ &= \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - \varepsilon\|u\|^p_{L^p(\Omega)} - \kappa(\varepsilon)\|u\|^{s+1}_{L^{s+1}(\Omega)} \\ &\geq \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - C_1\varepsilon\|u\|^p_{L^{p^*}(\Omega)} - \kappa(\varepsilon)C_2\|u\|^{s+1}_{L^{p^*}(\Omega)}. \end{aligned} \tag{5.55}$$

Moreover, by using Folland-Stein's continuous embedding $W \hookrightarrow L^{p^*}(\Omega)$ (Theorem 3.45), we have

$$I(u) \geq \frac{1}{p}\int_\Omega |\nabla_H u(x)|^p dx - C_1\varepsilon\left(\int_\Omega |u(x)|^{p^*}dx\right)^{\frac{p}{p*}} - C_2\kappa(\varepsilon)\left(\int_\Omega |u(x)|^{p*}dx\right)^{\frac{s+1}{p*}}$$
$$\geq \|u\|_W^p\left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\|u\|_W^{s+1-p}\right). \quad (5.56)$$

Assume that $u \in W$ and $\|u\|_W = \rho > 0$. From assumption $s+1 > p$, choosing ρ sufficiently small and choosing ε such that $\alpha := \rho^p\left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\rho^{s+1-p}\right) > 0$, we get

$$\inf_{u\in W, \|u\|_W=\rho} I(u) \geq \rho^p\left(\frac{1}{p} - C_1\varepsilon - C_2\kappa(\varepsilon)\rho^{s+1-p}\right) > 0. \quad (5.57)$$

Lemma 5.15 is proved. □

Now let us check the second assupmtion of the mountain pass theorem.

Lemma 5.16. *Assume that f be a Caratheódory function satisfying p1)-p4). Then there exists $v > 0$ a.e. in W, $\|v\|_W > \rho$ and $I(v) < \alpha$, where the constants α and ρ are given as in Lemma 5.15.*

Proof. By fixing $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{H}^n$ with $t > 0$. From Lemma 5.13, we calculate

$$I(tu) = \frac{1}{p}\int_\Omega |\nabla_H(tu)|^p dx - \int_\Omega F(x, tu(x))dx \leq \frac{t^p}{p}\int_\Omega |\nabla_H u|^p dx$$
$$- a_4 t^\mu \int_\Omega |u|^\mu dx + a_3|\Omega| = \frac{t^p}{p} - a_4 t^\mu \int_\Omega |u|^\mu dx + a_3|\Omega|. \quad (5.58)$$

From the assumption $\mu > p$ and by taking $t \to +\infty$, we have $I(tu) \to -\infty$. Consequently, by taking $v = \beta u$, with β sufficiently large, we obtain the desired result. □

From the above lemmas follow that the assumptions of the mountain pass theorem are fulfilled by the functional (5.48). Then we need to show the $(PS)_c$ compactness condition for the functional (5.48).

Lemma 5.17. *Assume that f be a Carathéodory function satisfying p1)-p4). Let $\{u_n\}$ be a sequence satisfying $I(u_n) \to c$ and*

$$\sup\{|\langle I'(u_n), \varphi\rangle| : \varphi \in W, \ \|\varphi\|_W = 1\} \to 0 \quad n \to \infty. \quad (5.59)$$

Then the sequence $\{u_n\} \subset W$ is bounded in W.

Proof. Assume that $\{u_n\} \subset W$ be a $(PS)_c$ sequence. Then for every $\varphi \in W$ we have

$$\langle I'(u_n), \varphi\rangle = \int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H \varphi dx - \int_\Omega f(x, u_n)\varphi dx, \quad (5.60)$$

and

$$I(u_n) = \frac{1}{p}\int_\Omega |\nabla_H u_n|^p dx - \int_\Omega F(x, u_n)dx. \quad (5.61)$$

Hence, we have

$$
\begin{aligned}
I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \int_\Omega \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx \\
&= \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \int_{\Omega\cap|u_n|\le r} \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx \\
&\qquad - \int_{\Omega\cap|u_n|>r} \left(F(x,u_n) - \frac{f(x,u_n)u_n}{\mu}\right) dx, \quad (5.62)
\end{aligned}
$$

where $\mu > p$.

Let us consider the second term on the right hand side. From Lemma 5.14 we calculate

$$
\begin{aligned}
&\left|\int_{\Omega\cap|u_n|\le r} F(x,u_n) - \frac{f(x,u_n)u_n}{\mu} dx\right| \\
&\qquad \le \left(\varepsilon r^p + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1})\right)|\Omega|. \quad (5.63)
\end{aligned}
$$

Let us denote the right hand side by

$$
\widetilde{\theta} := \left(\varepsilon r^p + \kappa(\varepsilon)r^{s+1} + \frac{1}{\mu}(\varepsilon r^2 + q\kappa(\varepsilon)r^{s+1})\right)|\Omega|
$$

By combining (5.63) and assumption $p3$), we get

$$
I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_\Omega |\nabla_H u_n|^p dx - \widetilde{\theta}. \quad (5.64)
$$

By the assumption in (5.59) with $\varphi := \frac{u_n}{\|u_n\|_W}$ for any n there exists a number $\lambda > 0$, such that

$$
\left|\left\langle I'(u_n), \left(\frac{u_n}{\|u_n\|_W}\right)\right\rangle\right| \le \lambda,
$$

with $I(u_n) \le \lambda$. Hence, we have

$$
I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \le \lambda(1 + \|u_n\|_W), \quad (5.65)
$$

combining this with (5.64) we arrive at

$$
\left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_W^p \le \lambda(1 + \|u_n\|_W) + \widetilde{\theta}.
$$

Finally,

$$
\begin{aligned}
\|u_n\|_W^p &\le \left(\frac{1}{p} - \frac{1}{\mu}\right)^{-1} (\lambda(1 + \|u_n\|_W) + \widetilde{\theta}) \\
&\le \left(\frac{1}{p} - \frac{1}{\mu}\right)^{-1} C_1(1 + \|u_n\|_W) \le C(1 + \|u_n\|_W).
\end{aligned}
$$

where C is a positive constant. □

Now we have to show that the $(PS)_c$ sequence of I has a strong convergent subsequence, so we can say I satisfies the $(PS)_c$ condition.

Lemma 5.18. *Under assumptions p1)-p4), if $\{u_n\} \subset W$ is a $(PS)_c$ sequence of I, then $\{u_n\}$ has a strong convergent subsequence in W.*

Proof. Since W is a Banach space, we have $u_n \rightharpoonup u$ weakly in W. Hence,

$$\begin{aligned}\langle I'(u_n), (u_n - u)\rangle = &\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H(u_n - u)dx \\ &- \int_\Omega f(x, u_n)(u_n - u)dx \to 0, \ \ n \to \infty.\end{aligned} \tag{5.66}$$

Also, we have $u_n \to u$ strongly convergence in $L^{s+1}(\Omega)$, where $s \in [p, p^* - 1)$. Then,

$$f(x, u_n)(u_n - u) \to 0, \ \text{ a.e. in } \Omega, \ n \to \infty. \tag{5.67}$$

Moreover, by using the Vitali convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)(u_n - u)dx = 0. \tag{5.68}$$

Plugging (5.68) in (5.66), we have

$$\int_\Omega |\nabla_H u_n|^{p-2}\nabla_H u_n \cdot \nabla_H(u_n - u)dx \to 0, \ \ n \to \infty. \tag{5.69}$$

Since $\{u_n\}$ weakly converges in W, we arrive at

$$\int_\Omega (|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u) \cdot \nabla_H(u_n - u)dx \to 0, \ \ n \to \infty. \tag{5.70}$$

Now let us give some useful vector inequalities. Let C_1, C_2 be positive constants depending only on p. Then, we have

$$|a - b|^p \le C_1(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b), \quad p \ge 2, \tag{5.71}$$

and

$$|a - b|^2 \le C_2(|a| + |b|)^{2-p}(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b), \quad 1 < p < 2, \tag{5.72}$$

for all vectors $a, b \in \mathbb{R}^N$. Firstly, let us consider the case $p \ge 2$. By applying (5.71) to (5.70), we have

$$\begin{aligned}\|u_n - u\|_W^p &= \int_\Omega |\nabla_H(u_n - u)|^p dx = \int_\Omega |\nabla_H u_n - \nabla_H u|^p dx \\ &\le C_1 \int_\Omega \left(|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u\right) \cdot (\nabla_H u_n - \nabla_H u)dx \\ &= C_1 \int_\Omega \left(|\nabla_H u_n|^{p-2}\nabla_H u_n - |\nabla_H u|^{p-2}\nabla_H u\right) \cdot \nabla_H(u_n - u)dx \to 0,\end{aligned} \tag{5.73}$$

as $n \to \infty$. It means for $p \ge 2$, we have

$$\|u_n - u\|_W \to 0, \quad n \to \infty.$$

Let us consider the case $1 < p < 2$. By using the fact $\{u_n\}$ is bounded in W, applying (5.70) to (5.72), we have

$$\begin{aligned}
\|u_n - u\|_W^p &= \int_\Omega |\nabla_H(u_n - u)|^p dx = \int_\Omega |\nabla_H u_n - \nabla_H u|^p dx \\
&\leq C_2 \left(\int_\Omega \left(|\nabla_H u_n|^{p-2} \nabla_H u_n - |\nabla_H u|^{p-2} \nabla_H u \right) \cdot (\nabla_H u_n - \nabla_H u) dx \right)^{\frac{p}{2}} \\
&\times \left(\int_\Omega |\nabla_H u_n| + |\nabla_H u| dx \right)^{\frac{(2-p)p}{2}} \\
&\leq C_3 \left(\int_\Omega \left(|\nabla_H u_n|^{p-2} \nabla_H u_n - |\nabla_H u|^{p-2} \nabla_H u \right) \cdot (\nabla_H u_n - \nabla_H u) dx \right)^{\frac{p}{2}} \\
&\to 0,
\end{aligned} \tag{5.74}$$

as $n \to \infty$. hence, we get

$$\|u_n - u\|_W \to 0, \quad n \to \infty, \quad 1 < p < \infty.$$

$\square$

Theorem 5.19. *Let f be a Carathéodory function satisfying p1)-p4). Then there exists a non-trivial weak solution of problem (5.44).*

Proof. By using Lemma 5.18, any $(PS)_c$ subsequence of I has strong convergence in W. Also, we have that

$$I(0) = 0,$$

and by taking ρ as in Lemma 5.16, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in W, \text{ and } \|u\|_W = \rho.$$

Therefore, now applying the mountain pass theorem, we get a critical point of the functional $I(u)$ which is a non-trivial weak solution of problem (5.44). $\square$

5.3.2. *On Stratified groups.* Then let us extend previous result on the case of stratified groups. Now let us consider the Dirichlet boundary value problem on stratified Lie groups $\mathbb{G}$:

$$\begin{cases} -\mathcal{L}_p u = f(x, u), \ x \in \Omega \subset \mathbb{G}, \ 1 < p < Q, \\ u(x) = 0, \ x \in \partial\Omega, \end{cases} \tag{5.75}$$

where f is a Carathéodory function satisfying the assumptions $p1) - p4)$ on $\mathbb{G}$. Then we have the following theorem:

Theorem 5.20. *There exists a non-trivial weak solution of problem (5.75).*

The proof is the same as the one of Theorem 5.19.

5.4. **Multiplicity of the weak solutions for the sub-Laplacian with Hardy potential.** In [80], Ghoussoub and Yuan considered the following problem with the Hardy-Sobolev potential:

$$\begin{cases} -\Delta_p u(x) - \lambda \frac{u(x)}{|x|^p} = |u|^{p-2}u, \; x \in \Omega \subset \mathbb{R}^n, \\ u(x) = 0, \;\; x \in \partial\Omega, \end{cases} \tag{5.76}$$

obtained existence and multiplicity of the weak solutions. Since then, analogues of the problem with the Hardy potential have been considered by many authors, see [81, 82, 83] and [84], for example.

In [85], Ghoussoub and Shakerian considered the following problem with fractional Laplacian and the Hardy-Sobolev potential:

$$(-\Delta_s)u - \gamma \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha}, \quad u > 0, \;\; x \in \mathbb{R}^n,$$

showed existence of the nontrivial weak solution. In this direction, most of studies have been dedicated to the single Hardy-Sobolev nonlinearity. In [86], the author investigated the following problem:

$$(-\Delta_s)u - \gamma \frac{u}{|x|^{2s}} = \frac{|u|^{2_s^*(\beta)-2}u}{|x|^\beta} + \frac{|u|^{2_s^*(\alpha)-2}u}{|x|^\alpha}, \quad u > 0, \;\; x \in \mathbb{R}^n,$$

showed multiplicity of the weak solution with the doubling Hardy-Sobolev potential, which generalises previous cases. In this section, we show multiplicity of the weak solutions with first stratum Hardy potential on Heisenberg and stratified groups.

5.4.1. *On Heisenberg group.* Let us recall the "horizontal" L^p-Caffarelli–Kohn–Nirenberg inequality on Heisenberg group.

Theorem 5.21 (Theorem 3.1., [29]). *For any $f \in C_0^\infty(\mathbb{H}^n \setminus \{z = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|2n-\gamma|}{p} \left\| \frac{f}{|z|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{H}^n)} \le \left\| \frac{\nabla_H f}{|z|^\alpha} \right\|_{L^p(\mathbb{H}^n)} \left\| \frac{f}{|z|^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{H}^n)}, \; \alpha, \beta \in \mathbb{R}, \tag{5.77}$$

where $\gamma = \alpha + \beta + 1$. If $\gamma \neq 2n$ then the constant $\frac{|2n-\gamma|}{p}$ is sharp.

When $\alpha = 0$ and $\beta = p - 1$, inequality (5.77) implies the first stratum Hardy inequality, that is, for all $f \in C_0^\infty(\mathbb{H}^n \setminus \{z = 0\})$, we have

$$\frac{|2n-p|}{p} \left\| \frac{f}{|z|} \right\|_{L^p(\mathbb{H}^n)} \le \|\nabla_H f\|_{L^p(\mathbb{H}^n)}, \;\; z = (x, y) \in \mathbb{R}^{2n}, \tag{5.78}$$

where $|z| = \sqrt{x_1^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_n^2}$.

Similarly with previous section, we also use the notation $W := S_0^{1,2}(\Omega)$. Also, let us define Sobolev space with the norm:

$$\|u\|_X := \left(\|\nabla_H u\|_W^2 - \lambda \int_\Omega \frac{|u|^2}{|z|^2} d\xi \right)^{\frac{1}{2}}, \quad 0 < \lambda < \overline{\lambda} = (n-1)^2 = \frac{(Q-4)^2}{4}. \tag{5.79}$$

Indeed, $\| \cdot \|_W$ and $\| \cdot \|_X$ are equivalent norms.

Let $\Omega \subset \mathbb{H}^n$ be a measurable set with sufficiently smooth boundary $\partial\Omega$ such that $(0,0,t) \notin \Omega$. Assume that $n > 1$ (that is, $Q > 4$), $0 < \lambda < \overline{\lambda} = (n-1)^2 = \frac{(Q-4)^2}{4}$ and $1 < p < 2^* - 1 = \frac{2Q}{Q-2} - 1$. In this subsection, we show multiplicity of positive weak solutions to the problem:

$$\begin{cases} -\Delta_H u(\xi) - \lambda \frac{u(\xi)}{|z|^2} = u^p(\xi), \ \xi \in \Omega \subset \mathbb{H}^n, \\ u(\xi) > 0, \ \ \xi \in \Omega, \\ u(\xi) = 0, \ \xi \in \partial\Omega, \end{cases} \tag{5.80}$$

where $|z| = \sqrt{x_1^2 + \ldots + x_n^2 + y_1^2 + \ldots + y_n^2}, \quad z = (x, y) \in \mathbb{R}^{2n}$.

To present a variational structure for problem (5.80), we introduce $I : W \to \mathbb{R}$ as follows

$$I(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|z|^2} d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi, \tag{5.81}$$

where $u_+ = \max\{u, 0\}$.

Note that I is a Fréchet differentiable functional with respect to $u \in W$ for any $\varphi \in W$, so we have

$$\langle I'(u), \varphi\rangle = \int_\Omega \nabla_H u \cdot \nabla_H \varphi d\xi - \lambda \int_\Omega \frac{u_+\varphi}{|z|^2} d\xi - \int_\Omega u_+^p \varphi d\xi. \tag{5.82}$$

For the functional I, let us verify the assumptions of Theorem 5.11.

Lemma 5.22. *Let Ω be a Haar measurable set in $\mathbb{H}^n$. Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$.*

Proof. Firstly, by the Folland-Stein-Sobolev inequality (Theorem 3.45), by using the facts that the norms (5.47) and (5.79) are equivalent, $2 < p+1 < 2^* = \frac{2Q}{Q-2}$ and $L^{2^*}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{L^{2^*}(\Omega)} \leq C\|u\|_W. \tag{5.83}$$

Now we give an estimate to the functional $I(u)$. So, using the above embedding and first stratum Hardy inequality we compute

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|z|^2} d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi \\ &\overset{(5.78)}{\geq} \frac{1}{2}\|u\|_W^2 - \frac{\lambda}{2\overline{\lambda}}\|u\|_W^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1} d\xi \\ &\overset{(5.83)}{\geq} \left(\frac{1}{2} - \frac{\lambda}{2\overline{\lambda}}\right)\|u\|_W^2 - \frac{C_2}{p+1}\|u\|_W^{p+1} \\ &= C_1\|u\|_W^2 - \frac{C_2}{p+1}\|u\|_W^{p+1}, \end{aligned} \tag{5.84}$$

where $C_1, C_2 > 0$. Let $u \in W$ and $\|u\|_W = \rho > 0$. By choosing ρ sufficiently small, we have $\alpha := \frac{C_1\rho^2}{2} - \frac{C_2\rho^{p+1}}{p+1} > 0$, thus, we arrive at

$$\inf_{u \in W, \|u\|_W = \rho} I(u) \geq \frac{C_1\rho^2}{2} - \frac{C_2\rho^{p+1}}{p+1} > 0. \tag{5.85}$$

$\square$

Lemma 5.23. *Under assumptions of Lemma 5.22, there exists $v > 0$ a.e. in W, $\|v\|_W > \rho$ and $I(v) < \alpha$, where the constants α and ρ are given as in Lemma 5.22.*

Proof. Let us fix $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{H}^n$ with $t > 0$. Then we get

$$\begin{aligned} I(tu) &= \frac{1}{2}\|tu\|_W^2 - \frac{\lambda}{2}\int_\Omega \frac{(tu_+)^2}{|z|^2}d\xi - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}d\xi \\ &\leq \frac{1}{2}\|tu\|_W^2 - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}d\xi \\ &= \frac{t^2}{2} - \frac{t^{p+1}}{p+1}\int_\Omega u_+^{p+1}d\xi. \end{aligned} \tag{5.86}$$

By the assumption $p > 1$ and by taking $t \to +\infty$, we get $I(tu) \to -\infty$. Thus, by setting $v = \beta u$, with β sufficiently large, we arrive at the desired result. $\square$

Finally, we need to check $(PS)_c$ condition for our functional. But we need to show some preliminary result.

Lemma 5.24. *Let $\{u_n\}$ be a bounded sequence in W such that $I'(u_n) \to 0$, as $n \to \infty$. Then there exists $u \in W$ such that, up to a subsequence, $\|u_n - u\|_W \to 0$, as $n \to \infty$.*

Proof. Since the norm $\|\cdot\|_W$ is equivalent to $\|\cdot\|_X$ and $\{u_n\}$ is a bounded in W with the norm $\|\cdot\|_W$, then we have

$$\|u_n\|_X = \|u_n\|_W - \lambda\int_\Omega \frac{u^2}{|z|^2}d\xi \overset{\lambda>0}{\leq} \|u_n\|_W \leq C. \tag{5.87}$$

By [3, Theorem 4.4.28], W is a Banach and reflexive space, so we have

$$u_n \rightharpoonup u, \text{ in } W, \text{ with the norm, } \|\cdot\|_X \tag{5.88}$$

and

$$u_n \to u, \text{ in } L^r(\mathbb{H}^n), 1 \leq r < 2^*, \;\; u_n \to u, \text{a.e. in } \mathbb{H}^\ltimes. \tag{5.89}$$

From this fact for $p+1 < 2^*$ and $I'(u_n) \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\|u_n\|_X^2 = \lim_{n\to\infty}\|(u_n)_+\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.90}$$

Similarly, we get

$$\lim_{n\to\infty}\left(\int_\Omega \nabla_H u \cdot \nabla_H u_n - \lambda\int_\Omega \frac{u(u_n)_+}{|z|^2}d\xi\right) = \|(u)_+\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.91}$$

By combining above facts, we obtain

$$\|u_n - u\|_X \to 0, \;\; n \to \infty.$$

By using property of norm's equivalence, we have

$$\|u_n - u\|_W \to 0, \;\; n \to \infty.$$

$\square$

Lemma 5.25. *Assume that $\{u_n\}$ be a $(PS)_c$ sequence such that Definition 5.10. Then there exists $u \in W$ such that*

$$\lim_{n\to\infty}\|u_n - u\|_W = 0. \tag{5.92}$$

Proof. By using Definition 5.10, we obtain

$$\begin{aligned} c+o(1) = I(u_n) - \frac{1}{p+1}\langle I'(u_n), u_n\rangle = \left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|_W^2 \\ -\lambda\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega \frac{(u_n)_+^2}{|z|^2}d\xi \geq \left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_n\|_W^2 \\ -C\|u_n\|_W^2 = C\|u_n\|_W^2, \end{aligned} \tag{5.93}$$

with $p+1>2$. Thus, we have $\|u_n\|_W \leq C$. Therefore, by Lemma 5.24, we have strong convergence of $\{u_n\}$ in W. □

Finally, let us give main result of this section.

Theorem 5.26. *Problem (5.80) has at least two positive solutions.*

Proof. Let us construct two solutions of the problem (5.80). By using Lemma 5.25, any $(PS)_c$ subsequence of I has strong convergence in W. Also, we have that

$$I(0)=0,$$

and by taking ρ as in Lemma 5.23, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in W, \text{ and } \|u\|_W = \rho.$$

Therefore, now applying the mountain pass theorem, we get a critical point of the functional $I(u)$ which is a positive weak solution of problem (5.80).

Now let us construct another solution of (5.80). By Lemma 5.22, there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_W = \rho$ and $I(u) \geq \alpha$ for all $u \in W$. Hence, we can choose

$$\rho_1 = \{\inf_{\rho\in\mathbb{R}} : I(u) > 0, \ \forall u \in W, \text{ with } \|u\|_X = \rho\}.$$

From this, we have $\rho_1 > 0$, then $I(u)$¡0. Assume that $\rho_2 > \rho_1$, s.t. $I(u)$ is a non-decreasing functional with $\rho_1 < \|u\|_X < \rho_2$. Then let us define the following smooth function $\theta(\eta)$ in the following form: $\theta(\eta)=1$ if $\eta \leq \rho_1$, and $\theta(\eta)=0$ if $\eta \geq \rho_2$.

We define the following energy functional:

$$I_2(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\lambda\theta(\|u\|_W)}{2}\int_\Omega \frac{u_+^2}{|z|^2}d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1}d\xi. \tag{5.94}$$

If $\|u\|_W \leq \rho_1$, then $I_2(u) = I(u)$ and $\|u\|_W \geq \rho_2$, so we have

$$I_2(u) := \frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{1}{p+1}\int_\Omega u_+^{p+1}d\xi.$$

It easy to see that I_2 is a coercive functional. By W is a Hilbert space, we have that the functional lower semi-continuity. Then we can say there exists minimum point of I_2 with negative energy, it means I_2 has a minimum point. It gives the second solution. □

5.4.2. *On stratified groups.* It is well-known fact, that the Heisenberg group is the most popular example of stratified groups. In this subsection extended results on stratified groups.

Let us give the L^p-Caffarelli–Kohn–Nirenberg inequality on stratified groups.

Theorem 5.27 (Theorem 3.1., [29]). *Let $\mathbb{G}$ be a stratified group with N_1 being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G}\setminus\{x'=0\})$, and all $1<p<\infty$, we have*

$$\frac{|N_1-\gamma|}{p}\left\|\frac{f}{|x'|^{\frac{\gamma}{p}}}\right\|_{L^p(\mathbb{G})} \leq \left\|\frac{\nabla_{\mathbb{G}} f}{|x'|^{\alpha}}\right\|_{L^p(\mathbb{G})}\left\|\frac{f}{|x'|^{\frac{\beta}{p-1}}}\right\|_{L^p(\mathbb{G})}, \tag{5.95}$$

where $\gamma=\alpha+\beta+1$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $\gamma=N_1$then the constant $\frac{|N_1-\gamma|}{p}$ is sharp.

If $\alpha=0$ and $\beta=p-1$, we obtain the first stratum Hardy inequality on $\mathbb{G}$.

Let $\Omega\subset\mathbb{G}$ be a measurable set with sufficiently smooth boundary $\partial\Omega$ such that $\{x'=0\}\notin\Omega$. Assume that dimension of the first stratum $N_1>2$, $0<\lambda<\overline{\lambda}=\frac{(N_1-2)^4}{2}$ and $1<p<2^*-1=\frac{2Q}{Q-2}-1$. Let us consider the following problem:

$$\begin{cases}\mathcal{L}u(x)-\lambda\frac{u(x)}{|x'|^2}=u^p(x),\ x\in\Omega\subset\mathbb{G},\\ u>0,\quad x\in\Omega,\\ u(x)=0,\ x\in\partial\Omega.\end{cases} \tag{5.96}$$

Theorem 5.28. *Problem (5.96) has at least two positive solutions.*

Proof. The proof follows the almost same lines of the proof of Theorem 5.26. Only difference is that now we use, that is, stratified group versions of Theorem 3.45 and Theorem 5.27 instead of Theorem 3.45 and Theorem 5.21, respectively. □

5.5. **Existence of the weak solution for the fractional sub-Laplacian with Hardy potential.** Let us continue our studying of the existence of the weak solution. In this section, we show existence of the weak solution for semilinear equation with fractional sub-Laplacian and Hardy potential. Since then, fractional analogues of this problem on Euclidean setting have been considered by many different authors, for example, in [87, 88, 89] and [90]. In addition, we refer to [91, 92] and [93] as well as references therein for fractional Laplacian problems with the Hardy potential.

Let us consider the following problem with Hardy potential on $\mathbb{G}$:

$$\begin{cases}(-\Delta_s)u(x)-\lambda\frac{u(x)}{|x|^{2s}}=u^p,\quad x\in\Omega\setminus\{0\}\subset\mathbb{G},\\ u(x)=0,\ x\in\mathbb{G}\setminus\Omega,\end{cases} \tag{5.97}$$

where Ω is an open bounded domain in $\mathbb{G}$ with smooth boundary, $0\leq\lambda<\overline{\lambda}$ is the best constant of the fractional Hardy inequality on $\mathbb{G}$, $1<p<2^*-1$ and $2s<Q$.

Setting $S:=W_0^{s,2}(\Omega)$, let us define the fractional Sobolev space on $\mathbb{G}$ with the norm

$$\|u\|_S^2:=[u]_{s,2}^2-\lambda\int_\Omega\frac{|u|^2}{|x|^{2s}}dx, \tag{5.98}$$

which is equivalent (by the fractional Hardy inequality) to the norm

$$\|u\|_{W_0^{s,2}(\Omega)} = [u]_{s,2}, \tag{5.99}$$

where $[\cdot]_{s,2} = [\cdot]_{s,2,\Omega}$ is the Gagliardo semi-norm which is defined in (2.14).

Definition 5.29. We say $u : \Omega \to \mathbb{R}$ is a weak solution of (5.97), if $u \in S$, such that

$$\int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+2s}}dxdy - \lambda \int_\Omega \frac{u_+\varphi}{|x|^{2s}}dx - \int_\Omega u_+^p \varphi dx = 0, \tag{5.100}$$

for all $\varphi \in S$, where $u_+ = \max\{u, 0\}$.

The energy functional corresponding to (5.97) can be given by the expression

$$I(u) = \frac{1}{2}\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|y^{-1}x|^{Q+2s}}dxdy - \lambda \int_\Omega \frac{(u_+)^2}{|x|^{2s}}dx\right) - \frac{1}{p+1}\int_\Omega u_+^{p+1}dx. \tag{5.101}$$

Note that I is a Fréchet differentiable functional with respect to $u \in S$ for any $\varphi \in S$, so we have

$$\begin{aligned}\langle I'(u), \varphi\rangle = \int_\Omega \int_\Omega \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|y^{-1}x|^{Q+2s}}dxdy - \lambda \int_\Omega \frac{u_+\varphi}{|x|^{2s}}dx \\ - \int_\Omega u_+^p \varphi dx.\end{aligned} \tag{5.102}$$

For the functional I, let us verify the assumptions of Theorem 5.11.

Lemma 5.30. *Let Ω be a Haar measurable set in $\mathbb{G}$. Then there exist positive constants $\rho, \alpha > 0$ such that $\|u\|_S = \rho$ and $I(u) \geq \alpha$ for all $u \in S$.*

Proof. Firstly, by Sobolev embedding theorem, by using the facts that the norms (5.98) and (5.99) are equivalent, $2 < p+1 < 2^* = \frac{2Q}{Q-2s}$ and $L^{2^*}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we have

$$\|u\|_{L^{p+1}(\Omega)} \leq C\|u\|_{L^{2^*}(\Omega)} \overset{(3.37)\ with\ \beta=0}{\leq} C\|u\|_S. \tag{5.103}$$

Now we give an estimate to the functional $I(u)$. So, using above embedding we compute

$$\begin{aligned} I(u) &= \frac{1}{2}\left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|y^{-1}x|^{Q+2s}}dxdy - \lambda \int_\Omega \frac{|u|^2}{|x|^{2s}}dx\right) - \frac{1}{p+1}\int_\Omega u_+^{p+1}dx \\ &= \frac{1}{2}\|u\|_S^2 - \frac{1}{p+1}\int_\Omega u_+^{p+1}dx \\ &\overset{(5.103)}{\geq} \frac{1}{2}\|u\|_S^2 - \frac{C}{p+1}\|u\|_S^{p+1}. \end{aligned} \tag{5.104}$$

Let $u \in W$ and $\|u\|_S = \rho > 0$. By choosing ρ sufficiently small, we have $\alpha := \frac{\rho^2}{2} - \frac{C\rho^{p+1}}{p+1} > 0$, thus, we arrive at

$$\inf_{u \in W, \|u\|_W = \rho} I(u) \geq \frac{\rho^2}{2} - \frac{C\rho^{p+1}}{p+1} > 0. \tag{5.105}$$

$\square$

Lemma 5.31. *Under assumptions of Lemma 5.30, there exists $0 < v \in S$ a.e. in W, $\|v\|_S > \rho$ and $I(v) < \alpha$, where the constants α and ρ are given as in Lemma 5.30.*

Proof. Let us fix $\|u\|_W = 1$ and $u \geq 0$ a.e. in $\mathbb{G}$ with $t > 0$. Then we calculate

$$I(tu) = \frac{1}{2}\|tu\|_S^2 - \frac{1}{p+1}\int_\Omega (tu_+)^{p+1}dx = \frac{t^2}{2} - \frac{t^{p+1}}{p+1}\int_\Omega u_+^{p+1}dx. \tag{5.106}$$

By the assumption $p > 1$ and by taking $t \to +\infty$, we get $I(tu) \to -\infty$. Thus, by setting $v = \beta u$, with β sufficiently large, we arrive at the desired result. $\square$

Lemma 5.32. *Let $\{u_n\}$ be a bounded sequence in S such that $I'(u_n) \to 0$, as $n \to \infty$. Then there exists $u \in S$ such that, up to a subsequence, $\|u_n - u\|_S \to 0$, as $n \to \infty$.*

Proof. Since the norm $\|\cdot\|_S$ is equivalent to $\|\cdot\|_{S_*}$, for the norm $\|\cdot\|_{S_*}$ there exists $u \in S$ and a subsequence $\{u_n\}$, such that,

$$u_n \rightharpoonup u, \text{ in } S, \tag{5.107}$$

and

$$u_n \to u, \text{ in } L^r(\mathbb{G}), 1 \leq r < 2^*, \ u_n \to u, \text{a.e. in } \mathbb{G}. \tag{5.108}$$

From this fact for $p + 1 < 2^*$ and $I'(u_n) \to 0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \|u_n\|_S^2 = \lim_{n\to\infty} \|u_n\|_{L^{p+1}(\Omega)}^{p+1} = \|u\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.109}$$

Similarly, we get

$$\lim_{n\to\infty}\left(\int_\Omega\int_\Omega \frac{(u_n(x) - u_n(y))(u(x) - u(y))}{|y^{-1}x|^{Q+2s}}dxdy - \lambda\int_\Omega \frac{u(u_n)_+}{|x|^{2s}}dx\right) = \|u\|_{L^{p+1}(\Omega)}^{p+1}. \tag{5.110}$$

Thus, we have

$$\|u_n - u\|_S \to 0, \ \ n \to \infty.$$

$\square$

Lemma 5.33. *Assume that $\{u_n\}$ be a $(PS)_c$ sequence such that Definition 5.10. Then there exists $u \in S$ such that*

$$\lim_{n\to\infty} \|u_n - u\|_S = 0. \tag{5.111}$$

Proof. By using Definition 5.10, we obtain

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{p+1}\langle I'(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{p+1}\right)[u_n]_{s,2}^2 \\ &- \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}dx \overset{(3.37) \text{ with } \beta=2s}{\geq} \left(\frac{1}{2} - \frac{1}{p+1}\right)[u_n]_{s,2}^2 \\ &- C[u_n]_{s,2}^2 = C[u_n]_{s,2}^2, \end{aligned} \tag{5.112}$$

with $p+1 > 2$. Thus, we have $\|u_n\|_S \leq C$. Therefore, by Lemma 5.32, we have strong convergence of $\{u_n\}$ in S. $\square$

We are now in a position to present the main result of this section.

Theorem 5.34. *Assume that $\Omega \subset \mathbb{G}$ be a Haar measurable set. Then there exists a non-trivial weak solution of problem (5.97).*

Proof. By using Lemma 5.33, any $(PS)_c$ subsequence of $I(u_n)$ has strong convergence in S. Also, we have that

$$I(0) = 0,$$

and by taking ρ as in Lemma 5.30, there exists α such that $I(u) \geq \alpha > 0 = I(0)$, where

$$u \in S, \text{ and } \|u\|_S = \rho.$$

Therefore, now applying the mountain pass theorem, we have a critical point of the functional $I(u)$ which is a non-trivial weak solution of problem (5.97). □

5.6. **Blow-up result to heat equation with fractional sub-Laplacian and logarithmic nonlinearity on homogeneous groups.** Firstly, heat equation with logrithmic nonlinearity with Cauchy-Dirichlet problem was considered in [94]:

$$\begin{cases} u_t(x,t) - \Delta_x u(x,t) = u \log |u|, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty). \end{cases} \tag{5.113}$$

Then they showed global solvability of solution by potential wells method. Also, they showed the following blow-up theorem (in the Euclidean setting):

Theorem 5.35. [94] *Assume that $u_0 \in H_0^1(\Omega)$ and*

$$J(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0|^2 dx - \frac{1}{2}\int_\Omega |u_0|^2 \log |u_0| dx + \frac{1}{4}\int_\Omega |u_0|^2 dx \leq M, \tag{5.114}$$

and

$$I(u_0) = \int_\Omega |\nabla u_0|^2 dx - \int_\Omega |u_0|^2 \log |u_0| dx < 0. \tag{5.115}$$

Then the weak solution of the problem (5.113) blows up at $+\infty$.

Moreover, in [95] it is showed the condition $J(u_0) \leq M$ is unnecessary to blow-up at infinity to a solution of the problem (5.113). In this section, we considered the heat equation with the fractional sub-Laplacian with logarithmic nonlinearity and we obtain the blow-up result. That is, we extend the blow-up theorem from [95] to general homogeneous groups.

Let us consider the following Cauchy-Dirichlet fractional heat equation on the homogeneous group:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + (-\Delta_s)u(x,t) = u(x,t)\log |u(x,t)|, & (x,t) \in \Omega \times (0,+\infty), \ \Omega \subset \mathbb{G}, \\ u(x,t) = 0, & (x,t) \in \mathbb{G} \setminus \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \end{cases} \tag{5.116}$$

where Δ_s is the fractional sub-Laplacian with $s \in (0,1)$.

For simplicity, we introduce the notations $H_0^s(\Omega) := W_0^{s,2}(\Omega)$ and $[u]_s := [u]_{s,2,\Omega}$. Let us give the definition of a weak solution.

Definition 5.36. Let $T > 0$. A function $u : \Omega \times [0,+\infty) \to \mathbb{R}$, $u = u(x,t) \in L^\infty(0,T; H_0^s(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$ is a called a weak solution of problem

(5.116) in $\Omega \times [0, +\infty)$, if $u_0 \in H_0^s(\Omega)$ and u satisfies (5.116) in the sense of distribution,

$$\int_\Omega u_t \varphi dx + \langle (-\Delta_s) u, \varphi \rangle = \int_\Omega u \log |u| \varphi dx, \tag{5.117}$$

for any $\varphi \in H_0^s(\Omega)$, $t \in (0, +\infty)$.

Let us introduce the definition of the blow-up in infinite time.

Definition 5.37. Let $u(x,t)$ be a weak solution of (5.116). We say that $u(x,t)$ blows up at $+\infty$ if

$$\lim_{t \to +\infty} \|u(\cdot, t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.118}$$

Let us consider the following energy functionals

$$J(u) = \frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2 \log |u| dx + \frac{1}{4}\int_\Omega |u|^2 dx, \tag{5.119}$$

and

$$I(u) = [u]_s^2 - \int_\Omega u^2 \log |u| dx. \tag{5.120}$$

By combining last facts, we have relation between two functionals in the following form:

$$J(u) = \frac{1}{2} I(u) + \frac{1}{4}\int_\Omega |u|^2 dx. \tag{5.121}$$

We have the following energy identity for (5.116).

Lemma 5.38. *Assume that u is a weak solution of the problem (5.116). Then we have*

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0), \quad \forall t \in (0, +\infty). \tag{5.122}$$

Proof. By taking inner product between (5.116) and u_t over Ω, we get

$$\int_\Omega |u_t|^2 dx + \langle (-\Delta_s) u, u_t \rangle = \int_\Omega u_t u \log |u| dx. \tag{5.123}$$

For the second term on the left hand side of (5.123), we have

$$\begin{aligned} \langle (-\Delta_s) u, u_t \rangle &= \int_\Omega \int_\Omega \frac{(u(x,t) - u(y,t))(u_t(x,t) - u_t(y,t))}{|y^{-1}x|^{Q+2s}} dxdy \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_\Omega \frac{|u(x,t) - u(y,t)|^2}{|y^{-1}x|^{Q+2s}} dxdy = \frac{1}{2}\frac{d[u]_s^2}{dt}. \end{aligned} \tag{5.124}$$

On the right hand side of (5.123), we get

$$\frac{du^2 \log |u|}{dt} = 2u_t u \log |u| + u u_t, \tag{5.125}$$

then

$$\begin{aligned} \int_\Omega u_t u \log |u| dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log |u| dx - \frac{1}{2}\int_\Omega u u_t dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \log |u| dx - \frac{1}{4}\frac{d}{dt}\int_\Omega u^2 dx. \end{aligned} \tag{5.126}$$

By using (5.124) and (5.126) in (5.123), we obtain

$$\int_\Omega |u_t|^2dx + \frac{d}{dt}\left(\frac{1}{2}[u]_s^2 - \frac{1}{2}\int_\Omega u^2\log|u|dx + \frac{1}{4}\int_\Omega u^2dx\right) = \int_\Omega |u_t|^2dx + \frac{d}{dt}J(u) = 0. \quad (5.127)$$

Integrating over $(0,t)$, we arrive at

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau + \int_0^t \frac{dJ(u)}{d\tau}d\tau = 0, \quad (5.128)$$

that is,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau + J(u) = J(u_0). \quad (5.129)$$

□

Now we are in the position to present the main result of this section.

Theorem 5.39. *Assume that u is a weak solution of (5.116) with $u_0 \in H_0^s(\Omega)$ and $I(u_0) < 0$. Then*

$$\lim_{t\to+\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \quad (5.130)$$

Proof. Firstly, by combining (5.117) with $u = \varphi$ we have

$$\frac{d}{dt}\|u\|^2_{L^2(\Omega)} = \frac{d}{dt}\int_\Omega u^2dx = 2\int_\Omega uu_tdx = -2\left(\langle(-\Delta_s)u,u\rangle - \int_\Omega u^2\log|u|dx\right) = -2I(u). \quad (5.131)$$

From last fact, (5.117) and (5.120), we get

$$\begin{aligned}\frac{dI(u)}{dt} &= \frac{d}{dt}\left([u]_s^2 - \int_\Omega u^2\log|u|dx\right)\\
&= 2\int_\Omega\int_\Omega \frac{(u(x,t)-u(y,t))(u_t(x,t)-u_t(y,t))}{|y^{-1}x|^{Q+2s}}dxdy\\
&- 2\int_\Omega u(x,t)u_t(x,t)\log|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx\\
&= 2\langle(-\Delta_s)u,u_t\rangle - 2\int_\Omega u(x,t)u_t(x,t)\log|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx\\
&= \int_\Omega |u_t(x,t)|^2dx - \int_\Omega u(x,t)u_t(x,t)dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} - \int_\Omega u(x,t)u_t(x,t)dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} + [u]_s^2 - \int_\Omega u^2(x,t)\log|u(x,t)|dx\\
&= -2\|u_t\|^2_{L^2(\Omega)} + I(u) \le I(u).\end{aligned} \quad (5.132)$$

Then by combining Grönwall–Bellman's inequality and $I(u_0) < 0$ in the last fact we have

$$I(u) \leq I(u_0)e^t \leq I(u_0) < 0, \ \ \forall t \in (0, +\infty). \tag{5.133}$$

It means that $I(u(x,t))$ is decreasing functional with respect to the argument t. By setting

$$A(t) = \int_0^t \|u\|^2_{L^2(\Omega)} dt, \ \ A'(t) = \|u\|^2_{L^2(\Omega)}, \tag{5.134}$$

and by Definition 5.36 we have

$$A''(t) = 2\int_\Omega u u_t dx = -2[u]_s + 2\int_\Omega u^2 \log|u| dx = -2I(u). \tag{5.135}$$

A simple calculation gives

$$(\log A(t))' = \frac{A'(t)}{A(t)}, \ \ (\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}. \tag{5.136}$$

Now let us estimate $\frac{A''(t)A(t)-(A'(t))^2}{A^2(t)}$. By using (5.134), (5.120) and Lemma 5.38, we obtain

$$A''(t) = -2I(u) = -4J(u) + A'(t) = -4J(u_0) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + A'(t). \tag{5.137}$$

Similarly, from (5.134) we obtain

$$\begin{aligned}
(A'(t))^2 &= \|u\|^4_{L^2(\Omega)} = \|u\|^4_{L^2(\Omega)} + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} \\
&+ \|u_0\|^4_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = \left(\int_\Omega \left(u^2 - u_0^2\right) dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} \\
&= \left(\int_\Omega \int_0^t \frac{\partial u^2}{\partial \tau} d\tau dx\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)} = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 \\
&\qquad + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}.
\end{aligned} \tag{5.138}$$

Finally, we obtain

$$(A'(t))^2 = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \tag{5.139}$$

It follows that

$$\begin{aligned}
A''(t)A(t) - (A'(t))^2 &= -4J(u_0)A(t) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau A(t) + A'(t)A(t) \\
&- 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} + \|u_0\|^4_{L^2(\Omega)} \\
&= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2\right) \\
&- 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)}.
\end{aligned} \tag{5.140}$$

By using the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\begin{aligned}
A''(t)A(t) - (A'(t))^2 &= 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2\right) \\
&\quad - 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)} A'(t) + \|u_0\|^4_{L^2(\Omega)} \\
&\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right).
\end{aligned} \tag{5.141}$$

By using (5.134), (5.135) and $I(u) \leq I(u_0) < 0$, we get

$$\begin{aligned}
&A'(t) = A'(0) - 2\int_0^t I(u(x,\tau))d\tau = -2I(u_0)t \geq 0, \quad t \geq 0, \\
&A(t) = -I(u_0)t^2 \geq 0, \quad t \geq 0.
\end{aligned} \tag{5.142}$$

By combining (5.142) and (5.120) in (5.141), we compute

$$\begin{aligned}
A''(t)A(t) - (A'(t))^2 &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right) \\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 4J(u_0)\right) \\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 2I(u_0) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{5.143}$$

From Definition 5.36, we have that $u_0 \in H_0^s(\Omega)$ and let

$$t > t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} \geq 0. \tag{5.144}$$

Firstly, let us consider the case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}. \tag{5.145}$$

By using this fact in (5.143), we get

$$\begin{aligned}
&A''(t)A(t) - (A'(t))^2 \\
&\qquad \geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad \geq A'(t)\left(\frac{-I(u_0)t_0^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = A'(t)\left(\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = A(t)\left(-I(u_0)(t_0+2) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad \geq A(t)\left(-I(u_0)\left(\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2 + 2\right) - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\qquad = 0.
\end{aligned} \tag{5.146}$$

Hence, we obtain

$$A''(t)A(t) - (A'(t))^2 \geq 0. \tag{5.147}$$

Similarly, in the other case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2,\ \frac{\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \tag{5.148}$$

we have

$$A''(t)A(t) - (A'(t))^2 \geq 0. \tag{5.149}$$

So we get

$$(\log A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}, \tag{5.150}$$

and integrating over (t_0, t), we have

$$(\log A(t))' - (\log A(t))'|_{t=t_0} = \int_{t_0}^{t} \frac{A''(\tau)A(\tau) - (A'(\tau))^2}{A^2(\tau)} d\tau \geq 0. \tag{5.151}$$

Hence, we have

$$(\log A(t))' \geq (\log A(t))'|_{t=t_0}. \tag{5.152}$$

Similarly, we have

$$\frac{A'(t_0)}{A(t_0)}(t-t_0) = (\log A(t))'|_{t=t_0}(t-t_0) \leq \int_{t_0}^{t} \log(A(\tau))' d\tau = \log(A(t)) - \log(A(t_0)). \tag{5.153}$$

Finally, we arrive at

$$A(t_0)e^{\frac{A'(t_0)}{A(t)}(t-t_0)} \leq A(t). \tag{5.154}$$

By summarising above facts (5.152)-(5.154) with $t \geq t_0$, we compute

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &= A'(t) = (\log A(t))'A(t) \geq (\log A(t))'|_{t=t_0} A(t) = \frac{A'(t_0)}{A(t_0)}A(t) = \frac{A(t)}{A(t_0)}A'(t_0) \\ &\geq A'(t_0)e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} = \|u(\cdot,t_0)\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)} \geq \|u_0\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}.\end{aligned} \tag{5.155}$$

That is,

$$\lim_{t\to+\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.156}$$

□

Remark 5.40. *In the Abelian (Euclidean) case $\mathbb{G} = (\mathbb{R}^N, +)$, we have $Q = N$ and $|\cdot| = |\cdot|_E$ ($|\cdot|_E$ is the Euclidean distance), if $s \to 1^-$ we get blow-up result at infinity in [94] and [95].*

5.7. **Non blow-up and blow-up results for the heat equation on stratified groups.** Similarly to previous section, we prove non blow-up and blow-up results for the heat equation on stratified groups.

Firstly, let us give Green's formulae which is play a key role in our proof.

Theorem 5.41 (Green's identity, [96])**.** *Let $Q \geq 3$ be a homogeneous dimension of a stratified group $\mathbb{G}$ and dx be the volume element on $\mathbb{G}$. Let $v \in C^1(\Omega) \cap C(\overline{\Omega})$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then the following Green's identity holds*

$$\int_\Omega \left((\widetilde{\nabla} v)u + v\Delta_{\mathbb{G}} u\right) dx = \int_{\partial\Omega} |u|^{p-2} v \langle \widetilde{\nabla} u, dx\rangle, \tag{5.157}$$

where

$$\widetilde{\nabla} u = \sum_{k=1}^{N_1} (X_k u)\, X_k.$$

In this section, we obtain a non-blow-up result for the following problem on stratified group:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \mu\Delta_{\mathbb{G}} u(x,t) = u(x,t)\ln|u(x,t)|, & (x,t) \in \Omega\times(0,T),\ \Omega\subset\mathbb{G}, \\ u(x,t)\Big|_{\partial\Omega} = 0, \quad t\in(0,T), \\ u(x,0) = u_0(x) \quad x\in\Omega, \end{cases} \tag{5.158}$$

where $\Delta_{\mathbb{G}}$ is the sub-Laplacian, μ is a positive constant and Ω is a bounded domain with smooth boundary.

Let us recall the definition of a weak solution.

Definition 5.42. Let $T > 0$. A function $u : \Omega \times [0, +\infty) \to \mathbb{R}$, $u = u(x,t) \in L^\infty(0,T; S^{1,2}_0(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))$ is a called a weak solution of problem (5.158) in $\Omega\times[0,+\infty)$, if $u_0 \in S^{1,2}_0(\Omega)$ and u satisfies (5.158) in the sense of distribution

$$\int_\Omega u_t\varphi dx - \mu\int_\Omega \varphi\Delta_{\mathbb{G}} u dx = \int_\Omega u\ln|u|\varphi dx, \tag{5.159}$$

for any $\varphi \in S^{1,2}_0(\Omega)$, $t\in(0,T)$.

Let us also recall the definition of blow-up at finite time.

Definition 5.43. Let $u(x,t)$ be a weak solution of (5.158). We say that $u(x,t)$ blows up at $T < +\infty$ if

$$\lim_{t \to T^-} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.160}$$

We use the following notations for energy functionals

$$J(u) = \frac{\mu}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \frac{1}{2}\int_\Omega u^2 \ln |u| dx + \frac{1}{4}\int_\Omega |u|^2 dx, \tag{5.161}$$

and

$$I(u) = \mu \int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \int_\Omega u^2 \ln |u| dx, \tag{5.162}$$

where $\mu > 0$. Thus, we have

$$J(u) = \frac{1}{2} I(u) + \frac{1}{4}\int_\Omega |u|^2 dx. \tag{5.163}$$

Also, one of the main tool is the logarithmic Sobolev-Follan-Stein inequality which is defined in Theorem 3.45.

Theorem 5.44. *Suppose that u is a weak solution of (5.158) with $u_0 \in S^{1,2}_0(\Omega)$ and $\mu \geq QC_S$, where C_S is the Sobolev-Folland-Stein constant. Then u does not blow-up at finite time.*

Proof. Let us define the following function:

$$A(t) := \int_0^t \|u(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau,$$

then we obtain

$$A'(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)},$$

$$A''(t) = -2I(u).$$

By using the logarithmic Sobolev-Folland-Sobolev inequality (Theorem 3.45) with $a = 1$, we get

$$
\begin{aligned}
A'(t)\ln A'(t) - A''(t) &= \|u\|^2_{L^2(\Omega)} \ln \|u\|^2_{L^2(\Omega)} + 2I(u) \\
&= 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + 2I(u) \\
&= 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + 2\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&\geq 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + \mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&\geq 2\|u\|^2_{L^2(\Omega)} \ln \|u\|_{L^2(\Omega)} + QC_S\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega u^2 \ln |u| dx \\
&= QC_S\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - 2\int_\Omega |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\Omega)}} dx \\
&\overset{(3.159)}{\geq} 2\int_{\mathbb{G}} |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\mathbb{G})}} dx + Q\|u\|^2_{L^2(\mathbb{G})} - 2\int_\Omega |u|^2 \ln \frac{|u|}{\|u\|_{L^2(\Omega)}} dx \\
&= Q\|u\|^2_{L^2(\mathbb{G})}.
\end{aligned}
\tag{5.164}
$$

It implies

$$A'(t)\ln A'(t) - A''(t) \geq Q\|u\|^2_{L^2(\mathbb{G})} \geq 0. \tag{5.165}$$

That is, $A'(t)\ln A'(t) \geq A''(t)$ which yields

$$\ln A'(t) \geq (\ln A'(t))'.$$

Now by integrating it over $(0,t)$, we obtain

$$\ln \|u(\cdot,t)\|^2_{L^2(\Omega)} = \ln A'(t) \leq e^t \ln A'(0) = e^t \ln \|u_0\|^2_{L^2(\Omega)}.$$

Finally, we arrive at

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u_0\|^{e^t}_{L^2(\Omega)}. \tag{5.166}$$

It means $\|u(\cdot,t)\|^2_{L^2(\Omega)}$ is bounded at finite time $T^* \in (0,\infty)$. □

Then let us show blow-up result in infinite time. Let us consider the following initial-boundary (Cauchy-Dirichlet) heat equation on stratified groups:

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} - \mu\Delta_{\mathbb{G}} u(x,t) = u(x,t)\ln|u(x,t)|, & (x,t) \in \Omega \times (0,+\infty),\ \Omega \subset \mathbb{G}, \\
u(x,t) = 0, \ \ (x,t) \in \Omega \times (0,+\infty), \\
u(x,0) = u_0(x),
\end{cases}
\tag{5.167}
$$

where $\Delta_{\mathbb{G}}$ is the sub-Laplacian and $\mu > 0$.

Definition 5.45. Assume that $u(x,t)$ be a weak solution of (5.167). We say that $u(x,t)$ blows up at $+\infty$ if

$$\lim_{t\to+\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.168}$$

We have the following energy identity for problem (5.167).

Lemma 5.46. *Suppose that u is a weak solution of the problem (5.167). Then we have*

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0), \ \ \forall t \in (0, +\infty), \tag{5.169}$$

where the functional J is defined by (5.161).

Proof. As usual, multiplying by u_t and integrating over Ω in (5.167), we get

$$\int_\Omega |u_t|^2 dx - \mu \int_\Omega \Delta_{\mathbb{G}} u \, u_t dx = \int_\Omega u_t u \ln |u| dx. \tag{5.170}$$

By using Green's identity to the second term on the left hand side of (5.170), we obtain

$$-\int_\Omega \Delta_{\mathbb{G}} u(x,t)\, u_t(x,t) dx = \int_\Omega \nabla_{\mathbb{G}} u(x,t) \cdot \nabla_{\mathbb{G}} u_t(x,t) dx = \frac{1}{2}\frac{d}{dt}\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)}. \tag{5.171}$$

On the other hand, we have

$$\frac{du^2 \ln |u|}{dt} = 2u_t u \ln |u| + u u_t, \tag{5.172}$$

that is,

$$\begin{aligned}\int_\Omega u_t u \ln |u| dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \ln |u| dx - \frac{1}{2}\int_\Omega u u_t dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \ln |u| dx - \frac{1}{4}\frac{d}{dt}\int_\Omega u^2 dx. \end{aligned} \tag{5.173}$$

By combining (5.171) and (5.173) with (5.170), we get

$$\begin{aligned}\int_\Omega |u_t|^2 dx + \frac{d}{dt}\left(\frac{\mu}{2}\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \frac{1}{2}\int_\Omega u^2 \ln |u| dx + \frac{1}{4}\int_\Omega u^2 dx\right) \\ = \int_\Omega |u_t|^2 dx + \frac{d}{dt} J(u) = 0. \end{aligned} \tag{5.174}$$

Now integrating over $(0,t)$, we arrive at

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{dJ(u)}{d\tau} d\tau = 0, \tag{5.175}$$

that is,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + J(u) = J(u_0). \tag{5.176}$$

□

Now we are in the position to present one of the main result of this section.

Theorem 5.47. *Assume that u be a weak solution of (5.167) with $u_0 \in S^{1,2}_0(\Omega)$ and $I(u_0) < 0$. Then*

$$\lim_{t \to +\infty} \|u(\cdot,t)\|^2_{L^2(\Omega)} = +\infty. \tag{5.177}$$

Proof. Firstly, by taking (5.159) with $u = \varphi$ we get

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} = \frac{d}{dt}\int_\Omega u^2 dx = 2\int_\Omega uu_t dx \\ = -2\left(\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2 \ln|u| dx\right) = -2I(u). \quad (5.178)\end{aligned}$$

By combining last fact with (5.159) and (5.162), we get

$$\begin{aligned}\frac{dI(u)}{dt} &= \frac{d}{dt}\left(\mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2 \ln|u| dx\right) \\ &= 2\mu\int_\Omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} u_t dx - 2\int_\Omega u(x,t)u_t(x,t)\ln|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\mu\int_\Omega u_t \Delta_{\mathbb{G}} u dx - 2\int_\Omega u(x,t)u_t(x,t)\ln|u(x,t)|dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\int_\Omega |u_t(x,t)|^2 dx - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} - \int_\Omega u(x,t)u_t(x,t)dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} + \mu\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} - \int_\Omega u^2(x,t)\ln|u(x,t)|dx \\ &= -2\|u_t\|^2_{L^2(\Omega)} + I(u) \leq I(u).\end{aligned} \quad (5.179)$$

From Grönwall–Bellman's inequality and $I(u_0) < 0$ we have

$$I(u) \leq I(u_0)e^t \leq I(u_0) < 0, \quad \forall t \in (0,T). \quad (5.180)$$

It shows that $I(u(x,t))$ is a decreasing functional with respect to t.

By setting

$$A(t) = \int_0^t \|u(\cdot,\tau)\|^2_{L^2(\Omega)} d\tau, \;\; A'(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)}, \quad (5.181)$$

and by Definition 5.159 we have

$$A''(t) = 2\int_\Omega uu_t dx = -2\|\nabla_{\mathbb{G}} u\|^2_{L^2(\Omega)} + 2\int_\Omega u^2 \ln|u| dx = -2I(u). \quad (5.182)$$

Now let us estimate

$$(\ln A(t))'' = \frac{A''(t)A(t) - (A'(t))^2}{A^2(t)}.$$

From (5.181), (5.162) and Lemma 5.46, we get

$$A''(t) = -2I(u) = -4J(u) + A'(t) = -4J(u_0) + 4\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + A'(t). \quad (5.183)$$

Similarly, from (5.181) we obtain

$$(A'(t))^2 = \|u\|^4_{L^2(\Omega)} + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} + \|u_0\|^4_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}$$
$$= 4\left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \quad (5.184)$$

Hence, we have

$$(A'(t))^2 = 4\left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u\|^2_{L^2(\Omega)}\|u_0\|^2_{L^2(\Omega)} - \|u_0\|^4_{L^2(\Omega)}. \quad (5.185)$$

It follows that

$$A''(t)A(t) - (A'(t))^2 = 4\left(\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau \int_0^t \|u\|^2_{L^2(\Omega)} - \left(\int_0^t\int_\Omega u_\tau u dx d\tau\right)^2\right)$$
$$- 4J(u_0)A(t) + A'(t)A(t) - 2\|u_0\|^2_{L^2(\Omega)}A'(t) + \|u_0\|^4_{L^2(\Omega)}. \quad (5.186)$$

From the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$A''(t)A(t) - (A'(t))^2 \geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right). \quad (5.187)$$

By using (5.181), (5.182) and $I(u) \leq I(u_0) < 0$, we have

$$\begin{aligned} &A'(t) = A'(0) - 2\int_0^t I(u(x,\tau))d\tau = -2I(u_0)t \geq 0, \quad t \geq 0,\\ &A(t) = -I(u_0)t^2 \geq 0, \quad t \geq 0. \end{aligned} \quad (5.188)$$

By using (5.188) and (5.162) in (5.187), we calculate

$$\begin{aligned} A''(t)A(t) - (A'(t))^2 &\geq A'(t)\left(\frac{A(t)}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(\frac{A'(t)}{2} - 4J(u_0)\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 4J(u_0)\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)t - 2I(u_0) - \|u_0\|^2_{L^2(\Omega)}\right)\\ &\geq A'(t)\left(\frac{-I(u_0)t^2}{2} - \|u_0\|^2_{L^2(\Omega)}\right) + A(t)\left(-I(u_0)(t+2) - \|u_0\|^2_{L^2(\Omega)}\right). \end{aligned} \quad (5.189)$$

From Definition 5.42, we have that $u_0 \in S_0^{1,2}(\Omega)$ and let

$$t > t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} \geq 0. \quad (5.190)$$

Let us consider the case

$$t_0 = \max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)} - 2, \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\} = \frac{\sqrt{2}\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}. \quad (5.191)$$

By combining this fact in (5.189), we obtain

$$\begin{aligned}
&A''(t)A(t)-(A'(t))^2\\
&\qquad\geq A'(t)\left(\frac{-I(u_0)t^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&\qquad\geq A'(t)\left(\frac{-I(u_0)t_0^2}{2}-\|u_0\|^2_{L^2(\Omega)}\right)+A(t)\left(-I(u_0)(t_0+2)-\|u_0\|^2_{L^2(\Omega)}\right)\\
&\qquad\geq A(t)\left(-I(u_0)\left(\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}\right)-\|u_0\|^2_{L^2(\Omega)}\right)=0.
\end{aligned}\tag{5.192}$$

Hence, we obtain

$$A''(t)A(t)-(A'(t))^2\geq 0.\tag{5.193}$$

Similarly, in the other case

$$t_0=\max\left\{\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\ \frac{\|u_0\|_{L^2(\Omega)}}{\sqrt{-I(u_0)}}\right\}=\frac{\|u_0\|^2_{L^2(\Omega)}}{-I(u_0)}-2,\tag{5.194}$$

we have

$$A''(t)A(t)-(A'(t))^2\geq 0.\tag{5.195}$$

So we have

$$(\ln A(t))''=\frac{A''(t)A(t)-(A'(t))^2}{A^2(t)},$$

and integrating over (t_0,t), we have

$$(\ln A(t))'-(\ln A(t))'|_{t=t_0}=\int_{t_0}^{t}\frac{A''(\tau)A(\tau)-(A'(\tau))^2}{A^2(\tau)}d\tau\geq 0.\tag{5.196}$$

Thus, we have

$$(\ln A(t))'\geq(\ln A(t))'|_{t=t_0}.\tag{5.197}$$

Similarly, we obtain

$$\frac{A'(t_0)}{A(t_0)}(t-t_0)=(\ln A(t))'|_{t=t_0}(t-t_0)\leq\int_{t_0}^{t}\ln(A(\tau))'d\tau=\ln(A(t))-\ln(A(t_0)).\tag{5.198}$$

Finally, we arrive at

$$A(t_0)e^{\frac{A'(t_0)}{A(t)}(t-t_0)}\leq A(t).\tag{5.199}$$

By using above facts (5.197)-(5.199) with $t\geq t_0$, we compute

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)}&=A'(t)=(\ln A(t))'A(t)\overset{(5.197)}{\geq}(\ln A(t))'|_{t=t_0}A(t)=\frac{A(t)}{A(t_0)}A'(t_0)\\
&\overset{(5.154)}{\geq}A'(t_0)e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}=\|u(\cdot,t_0)\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}\geq\|u_0\|^2_{L^2(\Omega)}e^{\frac{A'(t_0)}{A(t_0)}(t-t_0)}.
\end{aligned}\tag{5.200}$$

That is,

$$\lim_{t\to+\infty}\|u(\cdot,t)\|^2_{L^2(\Omega)}=+\infty.\tag{5.201}$$

$\square$

5.8. **Blow-up results for the viscoelastic equation on stratified groups.** The following viscoelastic wave equation with weak damping was considered by Messaoudi in [97].

$$\begin{cases} u_{tt} - \Delta u + \int_0^t k(t-\tau)\Delta u d\tau + a|u_t|^{q-2}u_t = |u|^{p-2}u, & (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{5.202}$$

where $u_0 \in W_0^{1,2}(\Omega)$, $u_1 \in L^2(\Omega)$ and $k \in C^1[0,T]$ satisfying $1 - \int_0^\infty k(\tau)d\tau = r > 0$. The author proved that any solution with negative initial energy $p > q$ blows up in finite-time and extended the result by considering positive initial energy in [98]. We refer [99] and [100] for the further discussions in this topic. Further, let us recall $L^p(\Omega)$-Poincaré inequality on stratified Lie groups (see [29]).

Theorem 5.48. *Assume that* $\Omega \subset \mathbb{G}$ *and* $f \in C_0^\infty(\Omega \setminus \{x' = 0\})$ *and* $R' = \sup_{x\in\Omega} |x'|$. *Then we have*

$$R\|f\|_p \leq \|\nabla_{\mathbb{G}} f\|_p, \quad 1 < p < \infty, \tag{5.203}$$

where $R = \frac{|N-p|}{R'p}$.

5.8.1. *Blow-up with strong damping.* In this subsection, we consider the following nonlinear viscoelastic wave equation on stratified Lie groups:

$$\begin{cases} u_{tt} - \Delta_{\mathbb{G}} u + \int_0^t k(t-\tau)\Delta_{\mathbb{G}} u d\tau - a\Delta_{\mathbb{G}} u_t = |u|^{p-2}u, & (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{5.204}$$

where $\Omega \subset \mathbb{G}$ is a Haar measurable set with a smooth boundary $\partial\Omega$, $N \geq 3$, where N is defined in (i), $u_0 \in S_0^{1,2}(\Omega)$, $u_1 \in L^2(\Omega)$, a is a positive constant and $p > 2$ satisfies the following condition.

$$\frac{2Q}{Q-2} > p > 2, \ \ Q \geq 3. \tag{5.205}$$

We assume that the function $C^1(0,\infty) \ni k : \mathbb{R}_+ \to \mathbb{R}_+$ has the following properties:

$$1 - \int_0^{+\infty} k(s)ds = r > \frac{1}{(p-1)^2} \tag{5.206}$$

and

$$k(s) \geq 0, \ k'(s) \leq 0. \tag{5.207}$$

Let us define the following functional

$$I(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) - \frac{1}{p}\|u(t)\|_p^p, \tag{5.208}$$

where $k \circ \nabla_{\mathbb{G}} u = \int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\cdot,t) - \nabla_{\mathbb{G}} u(\cdot,\tau)\|_2^2 d\tau$.

Let us give the main tools for obtaining blow-up result.

Lemma 5.49. *Assume that (5.206)-(5.207) hold true. Let u be a weak solution of (5.204), then we have*

(a) *$I(t)$ is a non-increasing function, i.e.,*

$$I'(t) \leq 0, \;\; \forall t \in [0,T]; \tag{5.209}$$

(b)

$$I(t) + a\int_0^t \|\nabla_{\mathbb{G}} u_t(\tau)\|^2 d\tau \leq I(0), \quad t \in [0,T], \;\; a > 0. \tag{5.210}$$

Proof. Let us rewrite the equation in (5.204) as follows

$$u_{tt} - \Delta_{\mathbb{G}} u + \int_0^t k(t-\tau)\Delta_{\mathbb{G}} u d\tau - a\Delta_{\mathbb{G}} u_t - |u|^{p-2}u = 0.$$

Multiplying both sides by u_t and integrating over Ω , we compute

$$\begin{aligned}
0 &= \int_\Omega u_{tt}u_t dx - \int_\Omega u_t \Delta_{\mathbb{G}} u dx + \int_0^t k(t-\tau)\int_\Omega u_t\Delta_{\mathbb{G}} u dx d\tau - a\int_\Omega u_t \Delta_{\mathbb{G}} u_t dx \\
&- \int_\Omega u_t |u|^{p-2} u dx \\
&\overset{(5.157)}{=} \int_\Omega u_{tt}u_t dx - \int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx - \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau \\
&+ a\int_\Omega |\nabla_{\mathbb{G}} u_t|^2 dx - \int_\Omega u_t|u|^{p-2}u dx \\
&= \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx\right) \\
&- \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau + a\int_\Omega |\nabla_{\mathbb{G}} u_t|^2 dx \\
&= \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla_{\mathbb{G}} u\|^2 - \frac{1}{p}\|u\|_p^p dx\right) - \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t \cdot \nabla_{\mathbb{G}} u dx d\tau \\
&+ a\|\nabla_{\mathbb{G}} u_t\|^2.
\end{aligned} \tag{5.211}$$

Let us calculate the following integral.

$$
\begin{aligned}
&\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t(t)\cdot\nabla_{\mathbb{G}} u(\tau)dxd\tau \\
&\qquad = \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t(t)\cdot(\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)+\nabla_{\mathbb{G}} u(t))dxd\tau \\
&\qquad = \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t(t)\cdot(\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t))dxd\tau \\
&\qquad + \int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t(t)\cdot\nabla_{\mathbb{G}} u(t)dxd\tau \\
&\qquad = -\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad + \frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad = -\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad + \frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(t)|^2 dxd\tau.
\end{aligned}
\tag{5.212}
$$

By direct calculation shows

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad = -\frac{1}{2}\frac{d}{dt}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)\|_2^2 d\tau \\
&\qquad + \frac{1}{2}\int_0^t k'(t-\tau)\int_\Omega |\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad = -\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}} u}{dt}+\frac{k'\circ\nabla_{\mathbb{G}} u}{2}
\end{aligned}
\tag{5.213}
$$

and

$$
\frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(t)|^2 dxd\tau = \frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}} u\|_2^2 d\tau\right)-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2. \tag{5.214}
$$

By changing the last expressions in (5.212), we have

$$
\begin{aligned}
&\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t(t)\cdot\nabla_{\mathbb{G}} u(\tau)dxd\tau \\
&\qquad= -\frac{1}{2}\int_0^t k(t-\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(\tau)-\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad+\frac{1}{2}\int_0^t k(\tau)\frac{d}{dt}\int_\Omega |\nabla_{\mathbb{G}} u(t)|^2 dxd\tau \\
&\qquad= -\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}} u}{dt}+\frac{k'\circ\nabla_{\mathbb{G}} u}{2}+\frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}} u\|^2 d\tau\right) \\
&\qquad-\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2.
\end{aligned}
\tag{5.215}
$$

Next, by using (5.215) in (5.211) yields

$$
\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla_{\mathbb{G}} u\|_2^2-\frac{1}{p}\|u\|_p^p dx\right)-\int_0^t k(t-\tau)\int_\Omega \nabla_{\mathbb{G}} u_t\cdot\nabla_{\mathbb{G}} u dxd\tau \\
&+a\|\nabla_{\mathbb{G}} u_t\|^2 \\
&= \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla_{\mathbb{G}} u\|^2-\frac{1}{p}\|u\|_p^p dx\right)+\frac{1}{2}\frac{dk\circ\nabla_{\mathbb{G}} u}{dt}-\frac{k'\circ\nabla_{\mathbb{G}} u}{2} \\
&-\frac{1}{2}\frac{d}{dt}\left(\int_0^t k(\tau)\|\nabla_{\mathbb{G}} u\|^2 d\tau\right)+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2+a\|\nabla_{\mathbb{G}} u_t\|^2 \\
&= \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla_{\mathbb{G}} u\|^2-\frac{1}{2}\int_0^t k(\tau)\|\nabla_{\mathbb{G}} u\|^2 d\tau+\frac{1}{2}k\circ\nabla_{\mathbb{G}} u-\frac{1}{p}\|u\|_p^p dx\right) \\
&+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2+a\|\nabla_{\mathbb{G}} u_t\|^2-\frac{k'\circ\nabla_{\mathbb{G}} u}{2} \\
&= \frac{dI}{dt}+\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|_2^2+a\|\nabla_{\mathbb{G}} u_t\|^2-\frac{k'\circ\nabla_{\mathbb{G}} u}{2},
\end{aligned}
\tag{5.216}
$$

that is,

$$
\begin{aligned}
\frac{dI}{dt} &= -\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2-a\|\nabla_{\mathbb{G}} u_t\|^2+\frac{k'\circ\nabla_{\mathbb{G}} u}{2} = -\frac{1}{2}k(t)\|\nabla_{\mathbb{G}} u\|^2 \\
&+\frac{1}{2}\int_0^t k'(t-\tau)\|\nabla_{\mathbb{G}} u(t)-\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau-a\|\nabla_{\mathbb{G}} u_t\|^2 \\
&\overset{(5.207)}{\leq} -a\|\nabla_{\mathbb{G}} u_t\|^2.
\end{aligned}
\tag{5.217}
$$

Hence, we get

$$
\frac{dI}{dt}\leq -a\|\nabla_{\mathbb{G}} u_t\|^2\leq 0,
\tag{5.218}
$$

that is,

$$
I'(t)\leq 0.
$$

It means we proved the statement (a). The part (b) follows from integrating (5.218) over $(0,t)$

$$I(t)-I(0)\leq -a\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau, \tag{5.219}$$

which is equivalent to

$$I(t)+a\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\leq I(0).$$

□

Now, we present the main result of this subsection.

Theorem 5.50. *Assume that $p>2$ satisfies (5.205), $a>0$ and $k\in C^1[0,T]$ satisfies the conditions (5.206) and (5.207). Let u be a solution of (5.204), satisfying*

$$\left(2(u,u_t)+a\|\nabla_{\mathbb{G}}u\|_2^2\right)|_{t=0}>\frac{2p}{\theta}I(0), \tag{5.220}$$

where $\theta=\max\limits_{\mu_1\in(0,1)}\theta(\mu_1)=\theta(\mu_1^*)$ with

$$\theta(\mu_1)=\min\left(((p+2)a\alpha\mu_1R)^{\frac{1}{2}},\frac{\alpha(1-\mu_1)}{a}\right). \tag{5.221}$$

Then, u blows up at a finite time.

Proof. Let us denote the following function:

$$Z(t)=2(u_t,u)+a\|\nabla_{\mathbb{G}}u(t)\|^2-\mu I(0), \tag{5.222}$$

where μ is a positive constant to be specified. By multiplying $u(t)$ the equation (5.204) and integrating over Ω, we have

$$(u_{tt},u)+a(\nabla_{\mathbb{G}}u,\nabla_{\mathbb{G}}u_t)=-\|\nabla_{\mathbb{G}}u\|^2-\int_0^t\int_\Omega k(t-\tau)\Delta_{\mathbb{G}}u(\tau)d\tau u(t)dx+\|u\|_p^p. \tag{5.223}$$

Then by using this fact, we get

$$\begin{aligned}Z'(t)&=2\|u_t\|^2+2(u_{tt},u)+2a(\nabla_{\mathbb{G}}u,\nabla_{\mathbb{G}}u_t)\\&=2\|u_t\|^2-2\|\nabla_{\mathbb{G}}u\|^2-2\int_\Omega k(t-\tau)\Delta_{\mathbb{G}}u(\tau)d\tau u(t)dx+2\|u\|_p^p.\end{aligned} \tag{5.224}$$

By using the first Green's identity, we compute

$$\begin{aligned}\int_0^t\int_\Omega k(t-\tau)u(t)\Delta_{\mathbb{G}}ud\tau dx&=-\int_0^t\int_\Omega k(t-\tau)(\nabla_{\mathbb{G}}u(t)\cdot\nabla_{\mathbb{G}}u(\tau))dxd\tau\\&=-\int_0^t\int_\Omega k(t-\tau)\nabla_{\mathbb{G}}u(t)\cdot(\nabla_{\mathbb{G}}(u(\tau)-u(t)))dxd\tau-\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}}u(t)\|^2d\tau\\&=-\int_0^t\int_\Omega k(t-\tau)\nabla_{\mathbb{G}}u(t)\cdot(\nabla_{\mathbb{G}}(u(\tau)-u(t)))dxd\tau-\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau\\&=-\int_0^t k(t-\tau)(\nabla_{\mathbb{G}}u(t),\nabla_{\mathbb{G}}(u(\tau)-u(t)))d\tau-\|\nabla_{\mathbb{G}}u(t)\|^2\int_0^t k(\tau)d\tau.\end{aligned} \tag{5.225}$$

This yields

$$
\begin{aligned}
Z'(t) &= 2\|u_t\|^2 + 2(u_{tt}, u) + 2a(\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} u_t) \\
&= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 - 2\int_{\Omega} k(t-\tau)\Delta_{\mathbb{G}} u(\tau) d\tau u(t) dx + 2\|u\|_p^p \\
&= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 + 2\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \\
&+ 2\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau + 2\|u\|_p^p.
\end{aligned} \tag{5.226}
$$

On the other hand, by using Young's inequality, we have

$$
\begin{aligned}
\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(t) - u(\tau))) d\tau &\leq \frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau \\
&+ \frac{1}{2p}\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau,
\end{aligned} \tag{5.227}
$$

that is,

$$
\begin{aligned}
&\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \geq \\
&-\frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau - \frac{1}{2p}\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau.
\end{aligned} \tag{5.228}
$$

Hence, in the view of (5.228), we have

$$
\begin{aligned}
Z'(t) &= 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 + 2\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t))) d\tau \\
&+ 2\|\nabla_{\mathbb{G}} u(t)\|^2 \int_0^t k(\tau) d\tau + \|u\|_p^p \\
&\geq 2\|u_t\|^2 - 2\|\nabla_{\mathbb{G}} u\|^2 - p\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau + 2\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau + 2\|u\|_p^p \\
&= (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- p\|u_t\|^2 - p\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 + 2\|u\|_p^p \\
&+ 2\left(-\frac{p}{2}\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u(\tau) - \nabla_{\mathbb{G}} u(t)\|^2 d\tau - \frac{1}{2p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau\right) \\
&\geq (p+2)\|u_t\|^2 + (p-2)\left(1 - \int_0^t k(\tau) d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau) d\tau - 2pI(t).
\end{aligned} \tag{5.229}
$$

By using the part (b) of Lemma 5.49 it follows that

$$
\begin{aligned}
Z'(t) &\geq (p+2)\|u_t\|^2 + (p-2)\left(1-\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau)d\tau - 2pI(t) \\
&\overset{(5.210)}{\geq} (p+2)\|u_t\|^2 + (p-2)\left(1-\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u(t)\|^2 \\
&- \frac{1}{p}\|\nabla_{\mathbb{G}} u\|^2 \int_0^t k(\tau)d\tau + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_\tau(\tau)\|^2 d\tau - 2pI(0) \\
&\overset{(5.206)}{\geq} (p+2)\|u_t\|^2 + \left((p-2)r - \frac{1}{p}(1-r)\right)\|\nabla_{\mathbb{G}} u\|^2 \\
&- 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t(\tau)\|^2 d\tau \\
&= (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{a>0}{\geq} (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0),
\end{aligned}
\tag{5.230}
$$

where $\alpha = \left((p-2)r - \frac{1}{p}(1-r)\right)$. Note that $\alpha > 0$ since the condition (5.206). Further, by using Young's inequality, we get

$$
2\left((p+2)a\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| \leq (p+2)\|u_t\|^2 + a\alpha\mu_1 R\|u\|^2. \tag{5.231}
$$

Combining Theorem 5.48 with this fact, we get

$$
\begin{aligned}
Z'(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&= (p+2)\|u_t\|^2 + \alpha a\mu_1\|\nabla_{\mathbb{G}} u\|^2 + (1-a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{(5.203)}{\geq} (p+2)\|u_t\|^2 + \alpha R a\mu_1\|u\|^2 + (1-a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{(5.231)}{\geq} 2\left((p+2)\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| + (1-a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&+ 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\
&\overset{a>0}{\geq} 2\left((p+2)a\alpha\mu_1 R\right)^{\frac{1}{2}} |(u_t, u)| + (1-a\mu_1)\alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \\
&\geq \theta(\mu_1)\left(2(u_t, u) + a\|\nabla_{\mathbb{G}} u\|^2 - \frac{2p}{\theta(\mu_1)} I(0)\right),
\end{aligned}
\tag{5.232}
$$

where R is defined in Theorem 5.48, $\mu_1 \in (0,1)$ is to be specified later and

$$\theta(\mu_1) = \min\left(((p+2)a\alpha\mu_1 R)^{\frac{1}{2}}, \frac{\alpha(1-\mu_1)}{a}\right). \tag{5.233}$$

Then we need to show that $K_1(\mu_1) = ((p+2)a\alpha\mu_1 R)^{\frac{1}{2}}$ is strictly increasing function for $\mu_1 \in [0,1]$ with $K_1(0) = 0$ and $K_1(1) = ((p+2)a\alpha R)^{\frac{1}{2}}$. Similarly, $K_2(\mu_2) = \frac{\alpha(1-\mu_1)}{a}$ is strictly decreasing function for $\mu_1 \in [0,1]$ with $K_2(0) = \frac{\alpha}{a}$ and $K_2(1) = 0$. Thus, $\theta(\mu_1)$ attains its maximum at the point $\mu_1 = \mu_1^*$, where μ_1^* is the root of the $((p+2)a\alpha\mu_1 R)^{\frac{1}{2}} = \frac{\alpha(1-\mu_1)}{a}$. Setting

$$\theta = \sup_{\mu_1 \in (0,1)} \theta(\mu_1) = \theta(\mu_1^*) \quad \text{and} \quad \mu = \frac{2p}{\theta}$$

in (5.222) implies that $Z(0) \geq 0$. Hence, we get

$$Z'(t) \geq \theta Z(t),$$

which implies

$$Z(t) \geq Z(0)\exp(\theta t),$$

that is,

$$Z(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

By introducing a new function

$$\xi(t) = \|u\|^2 + a\int_0^t \|\nabla_{\mathbb{G}} u(\tau)\|^2 d\tau + a(T-t)\|\nabla_{\mathbb{G}} u_0\|^2, \quad t \in [0,T], \tag{5.234}$$

we compute

$$\begin{aligned} \xi'(t) = 2(u_t, u) + a\|\nabla_{\mathbb{G}} u\|^2 - a\|\nabla_{\mathbb{G}} u_0\|^2 &= 2(u_t, u) + a\int_0^t \frac{d}{d\tau}\|\nabla_{\mathbb{G}} u\|^2 d\tau \\ &= 2(u_t, u) + 2a\int_0^t (\nabla_{\mathbb{G}} u_\tau(\tau), \nabla_{\mathbb{G}} u(\tau)) d\tau. \end{aligned} \tag{5.235}$$

It easy to see that $\xi''(t) = Z'(t)$, so we have

$$\begin{aligned} \xi''(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}} u_t\|^2 d\tau \\ &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0). \end{aligned} \tag{5.236}$$

Let $0 < \gamma < 1, \varepsilon > 0, T_B > 0$ be such that $\gamma(p+2) > 4 + \varepsilon = \nu$, and

$$(p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2 - 2pI(0) \geq \gamma((p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}} u\|^2) \quad t > T_B. \tag{5.237}$$

Thus, by using these facts, we have

$$\begin{aligned}\xi''(t) &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}}u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \\ &\geq (p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}}u\|^2 - 2pI(0) + 2ap\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \\ &\geq \gamma((p+2)\|u_t\|^2 + \alpha\|\nabla_{\mathbb{G}}u\|^2) + a\gamma(p+2)\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \\ &\geq (4+\varepsilon)(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau) \\ &= \nu(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau), \quad t > T_B.\end{aligned} \tag{5.238}$$

Next, from Cauchy-Schwarz-Bunyakovsky inequality yields the following estimates:

$$|(u_t, u)| \leq \|u_t\|^2\|u\|^2, \tag{5.239}$$

$$\left(\int_0^t (\nabla_{\mathbb{G}}u_t, \nabla_{\mathbb{G}}u)d\tau\right)^2 \leq \int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \int_0^t \|\nabla_{\mathbb{G}}u\|^2 d\tau. \tag{5.240}$$

Hence, we get

$$\begin{aligned}&2(u_t,u)\int_0^t (\nabla_{\mathbb{G}}u_t, \nabla_{\mathbb{G}}u)d\tau \\ &\overset{(5.239),(5.240)}{\leq} 2\|u_t\|\|u\|\left(\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\nabla_{\mathbb{G}}u\|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq \|u\|^2\left(\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau\right) + \|u_t\|^2\left(\int_0^t \|\nabla_{\mathbb{G}}u\|^2 d\tau\right).\end{aligned} \tag{5.241}$$

Hence, for $t > T_B$ we get

$$\begin{aligned}\xi''(t)\xi(t) - \frac{\nu}{4}(\xi'(t))^2 &> \nu\left(\|u_t\|^2 + a\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau\right)\left(\|u\|^2 + a\int_0^t \|\nabla_{\mathbb{G}}u(\tau)\|^2 d\tau\right) \\ &- \nu\left(2(u_t,u) + 2a\int_0^t (\nabla_{\mathbb{G}}u_t(\tau), \nabla_{\mathbb{G}}u(\tau))d\tau\right)^2 \\ &= \nu\Big(\|u_t\|^2\|u\|^2 + a\|u_t\|^2\int_0^t \|\nabla_{\mathbb{G}}u(\tau)\|^2 d\tau + a\|u\|^2\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau \\ &+ a^2\int_0^t \|\nabla_{\mathbb{G}}u_t\|^2 d\tau\int_0^t \|\nabla_{\mathbb{G}}u(\tau)\|^2 d\tau\Big)\end{aligned}$$

$$
\begin{aligned}
&-\nu\Big((u_t,u)^2+2a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\\
&\quad+a^2\left(\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\right)^2\Big)\\
&\overset{(5.239)}{\geq}\ \nu\Big(a\|u_t\|^2\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+a\|u\|^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\\
&\quad+a^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau\Big)\\
&\quad-\gamma\Big(2a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\\
&\quad+a^2\left(\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\right)^2\Big)\\
&\overset{(5.240)}{\geq}\ a\nu\left(\|u_t\|^2\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+\|u\|^2\int_0^t\|\nabla_{\mathbb{G}}u_t\|^2d\tau\right)\\
&\quad-2\nu a(u_t,u)\int_0^t(\nabla_{\mathbb{G}}u_t(\tau),\nabla_{\mathbb{G}}u(\tau))d\tau\\
&\overset{(5.241)}{\geq}\ 0.
\end{aligned}
\tag{5.242}
$$

By setting $\phi(s)=\xi(t-T_B)$, where $s=t-T_B$, it is easy to see that

$$\phi''\phi-\frac{\gamma}{4}(\phi')^2\geq 0.$$

Thus, there exists $T_B<t<T$ such that

$$\lim_{t\to T_B}\phi(s)=+\infty, \tag{5.243}$$

i.e.,

$$\lim_{t\to T_B}\left(\|u\|^2+a\int_0^t\|\nabla_{\mathbb{G}}u(\tau)\|^2d\tau+(T-t)\|\nabla_{\mathbb{G}}u_0\|^2\right)=+\infty. \tag{5.244}$$

Hence, in the view of the last expression we have

$$\|\nabla_{\mathbb{G}}u\|^2\to+\infty,\quad t\to T_B.$$

□

5.8.2. *Blow-up with weak damping.* In this subsection, we consider the viscoelastic wave equation with weak damping for the sub-Laplacian:

$$
\begin{cases}
u_{tt}-\Delta_{\mathbb{G}}u+\int_0^t k(t-\tau)\Delta_{\mathbb{G}}u d\tau+a|u_t|^{q-2}u_t=|u|^{p-2}u, & (x,t)\in\Omega\times[0,T],\\
u(x,t)=0, \quad x\in\partial\Omega,\\
u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),
\end{cases}
\tag{5.245}
$$

where $\Omega\subset\mathbb{G}$, is a Haar measurable set with a smooth boundary $\partial\Omega$, $a>0$, $p>2$ $q\geq 1$, $u_0\in S_0^{1,2}(\Omega)$, and $u_1\in L^2(\Omega)$. The function $I(t)$ is defined as in (5.208) and the function k satisfies (5.206)-(5.207). Further, let p and q be such that

$$\max\{p,q\} \leq \frac{2(Q-1)}{Q-2}. \tag{5.246}$$

We state the following lemmas which will be useful in proving blow-up result for (5.245).

Lemma 5.51. *Assume that p, q satisfy (5.246). Then, we have*

$$\|u\|_p^\gamma \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right), \quad 2 \leq \gamma \leq p, \tag{5.247}$$

where C is a positive constant which depends only on the Haar measure of Ω.

Proof. Suppose that $\|u\|_p > 1$. Since $2 \leq \gamma \leq p$, Sobolev Embedding Theorem 3.45 with $2^* = \frac{2Q}{Q-2}$ yields

$$\|u\|_p^\gamma \leq \|u\|_p^p \leq \|u\|_p^p + \|u\|_{2^*}^2 \leq \|u\|_p^p + C\|\nabla_{\mathbb{G}} u\|_2^2 \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right). \tag{5.248}$$

Now suppose $\|u\|_p \leq 1$. Let $p = \frac{Qp'}{Q-p'}$ with $1 < p' < Q$. Then we have the $1 < p' < p$ yielding continuous embedding, i.e., $L^p(\Omega) \hookrightarrow L^{p'}(\Omega)$. Hence, we have

$$\|\nabla_{\mathbb{G}} u\|_{p'} \leq C\|\nabla_{\mathbb{G}} u\|_p. \tag{5.249}$$

Since $2 \leq \gamma$, we get

$$\begin{aligned} \|u\|_p^\gamma \leq \|u\|_p^2 \leq C\|\nabla_{\mathbb{G}} u\|_{p'}^2 &\overset{(5.249)}{\leq} C\|\nabla_{\mathbb{G}} u\|_p^2 \leq C\|\nabla_{\mathbb{G}} u\|_p^2 + \|u\|_p^p \\ &\leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right). \end{aligned} \tag{5.250}$$

$\square$

Lemma 5.52. *Assume that u be a weak solution of (5.245) with (5.246). Then we get*

$$\|u\|_p^\gamma \leq C\left(I(t) - \|u_t\|^2 - (k \circ \nabla_{\mathbb{G}} u) + \|u\|_p^p\right), \quad \forall t \in [0,T], \tag{5.251}$$

where $2 \leq \gamma \leq p$ and C is a positive constant.

Proof. The function $I(t)$ is given by

$$I(t) = \frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) - \frac{1}{p}\|u(t)\|_p^p. \tag{5.252}$$

Therefore, by combining (5.206) and (5.207), we compute

$$\begin{aligned} r\|\nabla_{\mathbb{G}} u(t)\|_2^2 &\overset{(5.206)}{=} \left(1 - \int_0^\infty k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 \\ &\overset{(5.207)}{\leq} \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 \\ &= 2I(t) - \|u_t(t)\|_2^2 - k \circ \nabla_{\mathbb{G}} u + \frac{2}{p}\|u\|_p^p. \end{aligned} \tag{5.253}$$

Now we apply Lemma 5.51 with $2 \leq \gamma \leq p$, to obtain

$$\|u\|_p^\gamma \leq C\left(\|u\|_p^p + \|\nabla_{\mathbb{G}} u\|^2\right) \overset{(5.253)}{\leq} C\left(I(t) - \|u_t\|^2 - (k \circ \nabla_{\mathbb{G}} u) + \|u\|_p^p\right). \tag{5.254}$$

$\square$

Lemma 5.53. *Assume that (5.206)-(5.207) are satisfied. Suppose u be a weak solution of (5.245), then $I(t)$ is a non-increasing function for $t \in [0,T]$, i.e.,*

$$I'(t) \leq 0, \quad \forall t \in [0,T]. \tag{5.255}$$

We omit the proof of Lemma 5.53 since it is similar to that of Lemma 5.49. The main result of this section is the following theorem.

Theorem 5.54. *Suppose that $q > 1$ and $p > \max\{2, q\}$ satisfy the condition (5.246). If (5.206) and (5.207) hold with $I(0) < 0$, then solution u of (5.245) blows up at a finite time.*

Proof. From Lemma 5.53, we have

$$I'(t) \leq 0, \tag{5.256}$$

therefore,

$$I(t) \leq I(0), \quad \forall t \in [0,T].$$

Let us denote by $Z(t) = -I(t)$. Then, we have

$$\begin{aligned} 0 < Z(0) &\leq Z(t) = -I(t) \\ &= -\frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) + \frac{1}{p}\|u(t)\|_p^p \\ &= -\frac{1}{2}\|u_t(t)\|_2^2 - \frac{1}{2}\left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 - \frac{1}{2} k \circ \nabla_{\mathbb{G}} u + \frac{1}{p}\|u(t)\|_p^p \\ &\overset{(5.206),(5.207)}{\leq} \frac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{5.257}$$

Similarly by Lemma 5.49, we get

$$Z'(t) = -I'(t) = a\|u_t\|_q^q - \frac{1}{2}(k' \cdot \nabla_{\mathbb{G}} u) + \frac{1}{2} k(t)\|\nabla_{\mathbb{G}} u\|^2 \overset{(5.206),(5.207)}{\geq} 0. \tag{5.258}$$

Let us also define the following function

$$A(t) = Z^{1-\beta}(t) - \varepsilon(u_t, u), \tag{5.259}$$

where $0 < \beta \leq \min\{\frac{p-2}{2p}, \frac{p-q}{p(q-1)}\}$. By means of direct calculations and the Cauchy-Bunyakovsy-Schwarz inequality, we have

$$\begin{aligned}
A'(t) &= (1-\beta)Z^{-\beta}(t)Z'(t) - \varepsilon(u_{tt}, u) - \varepsilon\|u_t\|^2 \\
&= (1-\beta)Z^{-\beta}(t)Z'(t) - \varepsilon\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\int_0^t k(t-\tau)(\nabla_{\mathbb{G}} u(\tau), \nabla_{\mathbb{G}} u(t))d\tau \\
&+ \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&\overset{(5.225),(5.258)}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \varepsilon\int_0^t\int_\Omega k(t-\tau)(\nabla_{\mathbb{G}} u(t), \nabla_{\mathbb{G}}(u(\tau) - u(t)))dxd\tau \\
&+ \varepsilon\|\nabla_{\mathbb{G}} u(t)\|^2\int_0^t k(\tau)d\tau + \varepsilon\|u_t\|^2 \\
&\overset{C-B-S}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u\|^2\|\nabla_{\mathbb{G}}(u(\tau) - u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}} u(t)\|^2\int_0^t k(\tau)d\tau.
\end{aligned} \tag{5.260}$$

In the view of (5.208), we get

$$\frac{1}{p}\|u\|_p^p = Z(t) + \frac{1}{2}\left(\|u_t(t)\|^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|^2 + k \circ \nabla_{\mathbb{G}} u\right). \tag{5.261}$$

On the other hand, by combining (5.260) with (5.228), we have

$$\begin{aligned}
A'(t) &\geq a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\|u\|_p^p - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u\|^2\|\nabla_{\mathbb{G}}(u(\tau) - u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}} u(t)\|^2\int_0^t k(\tau)d\tau \\
&\overset{(5.261)}{=} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q - \varepsilon\|\nabla_{\mathbb{G}} u\|^2 - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx + \varepsilon\|u_t\|^2 \\
&+ \frac{\varepsilon p}{2}\left(2Z(t) + \|u_t\|^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u\|^2 + k \circ \nabla_{\mathbb{G}} u\right) \\
&- \varepsilon\int_0^t k(t-\tau)\|\nabla_{\mathbb{G}} u\|^2\|\nabla_{\mathbb{G}}(u(\tau) - u(t))\|^2 d\tau + \varepsilon\|\nabla_{\mathbb{G}} u(t)\|^2\int_0^t k(\tau)d\tau \\
&\overset{(5.228)}{\geq} a(1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon\int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \left(\frac{\varepsilon p}{2} - \varepsilon\delta\right)(k \circ \nabla_{\mathbb{G}} u) + \left(\left(\frac{p}{2} - 1\right) - \varepsilon\left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u\|^2,
\end{aligned} \tag{5.262}$$

where $\delta \in (0, \frac{p}{2})$. By applying Young's inequality to estimate the fourth term on the right hand side of the (5.262) to obtain

$$
\begin{aligned}
A'(t) &\geq (1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon \int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \left(\frac{\varepsilon p}{2} - \varepsilon\delta\right)(k \circ \nabla_{\mathbb{G}} u) + \left(\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau\right)\|\nabla_{\mathbb{G}} u\|^2 \\
&= (1-\beta)Z^{-\beta}(t)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon p Z(t) - a\varepsilon \int_\Omega |u_t|^{q-2}u_t u dx \\
&+ \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) + \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon p Z(t) \\
&\geq a\left((1-\beta)Z^{-\beta} - \frac{\varepsilon \lambda^{-q'}}{q'}\right)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 - \frac{\varepsilon a \lambda^q}{q}\|u\|_q^q + \varepsilon p Z(t), \quad \forall \lambda > 0,
\end{aligned}
\tag{5.263}
$$

where

$$
C_1 = \frac{\varepsilon p}{2} - \varepsilon\delta > 0, \quad C_2 = \left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t k(\tau)d\tau > 0.
$$

Then, by setting $\lambda^{-q'} = \chi Z^{-\beta}(t)$ we get

$$
\begin{aligned}
A'(t) &\geq a\left((1-\beta)Z^{-\beta} - \frac{\varepsilon \lambda^{-q'}}{q'}\right)\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 \\
&+ \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) + \varepsilon p Z(t) + \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 - \frac{\varepsilon a \lambda^q}{q}\|u\|_q^q \\
&= a\left((1-\beta) - \frac{\varepsilon \chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - \frac{\varepsilon a \chi^{1-q}}{q} Z^{-\beta(1-q)}\|u\|_q^q\right).
\end{aligned}
\tag{5.264}
$$

Next, from (5.257) and the fact that $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for $p > q$, we have

$$
\|u\|_q^q \leq C\left(\frac{1}{p}\right)^{\beta(q-1)} \|u\|_p^{q+\beta p(q-1)}. \tag{5.265}
$$

The last inequality applied to (5.264) yields

$$
\begin{aligned}
A'(t) &\geq a\left((1-\beta) - \frac{\varepsilon \chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - \frac{a \chi^{1-q}}{q}\|u\|_q^q\right) \\
&\geq a\left((1-\beta) - \frac{\varepsilon \chi}{q'}\right)Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&+ \varepsilon C_2 \|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - C\frac{a \chi^{1-q}}{q}\left(\frac{1}{p}\right)^{\beta(q-1)}\|u\|_p^{q+\beta p(q-1)}\right).
\end{aligned}
\tag{5.266}
$$

Now, by applying Lemma 5.52 with $\gamma = q + \beta p(q-1) \leq p$.

$$
\begin{aligned}
A'(t) &\geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&\quad + \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) - C\frac{a\chi^{1-q}}{q}\left(\frac{1}{p}\right)^{\beta(q-1)}\|u\|_p^\gamma\right) \\
&\overset{(5.251)}{\geq} a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \left(\varepsilon + \frac{\varepsilon p}{2}\right)\|u_t\|^2 + \varepsilon C_1(k \circ \nabla_{\mathbb{G}} u) \\
&\quad + \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(pZ(t) + C_1'\chi^{1-q}\left(Z(t) + \|u_t\|^2 + (k \circ \nabla_{\mathbb{G}} u) - \|u\|_p^p\right)\right) \\
&\geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \varepsilon\left(\frac{p}{2} + 1 + C_1\chi^{1-q}\right)\|u_t\|^2 - \varepsilon C_1'\chi^{1-q}\|u\|_p^p \\
&\quad + \varepsilon\left(C_1 + C_1'\chi^{1-q}\right)(k \circ \nabla_{\mathbb{G}} u) + \varepsilon C_2\|\nabla_{\mathbb{G}} u\|^2 + \varepsilon\left(p + C_1'\chi^{1-q}\right) Z(t),
\end{aligned}
\tag{5.267}
$$

where $C_1' = \frac{aC\left(\frac{1}{p}\right)^{\beta(q-1)}}{q}$. From assumption $I(t) < 0$, that is,

$$
Z(t) \geq -\frac{1}{2}\left(\|u_t(t)\|_2^2 + \left(1 - \int_0^t k(s)ds\right)\|\nabla_{\mathbb{G}} u(t)\|_2^2 + k \circ \nabla_{\mathbb{G}} u\right) + \frac{1}{p}\|u(t)\|_p^p. \tag{5.268}
$$

By setting $p = 2b + (p - 2b)$ where $b = \min\{C_1, C_2\}$ and letting χ to be large enough in (5.267) we have

$$
A'(t) \geq a\left((1-\beta) - \frac{\varepsilon\chi}{q'}\right) Z^{-\beta}\|u_t\|_q^q + \varepsilon\sigma\left(Z(t) + \|u_t\|^2 + \|u\|_p^p + k \circ \nabla_{\mathbb{G}} u\right), \tag{5.269}
$$

where $\sigma > 0$. Next, we choose sufficiently small ε so that $(1-\beta) - \frac{\varepsilon\chi}{q'} > 0$. Thus, we have

$$
A'(t) > \varepsilon\sigma\left(Z(t) + \|u_t\|^2 + \|u\|_p^p + k \circ \nabla_{\mathbb{G}} u\right), \tag{5.270}
$$

and

$$
A(0) = Z^{1-\beta}(0) + \varepsilon(u_0, u_1) > 0.
$$

Hence,

$$
0 < A(0) \leq A(t), \quad \forall t \in [0, T].
$$

Now, by using the Cauchy-Bunyakovsky-Schwarz inequality, embedding of spaces and Young's inequalities, we have

$$
|(u_t, u)|^{\frac{1}{1-\beta}} \leq \|u_t\|^{\frac{1}{1-\beta}}\|u\|^{\frac{1}{1-\beta}} \leq C\|u_t\|^{\frac{1}{1-\beta}}\|u\|_p^{\frac{1}{1-\beta}} \leq C\left(\|u\|_p^\gamma + \|u_t\|^2\right), \tag{5.271}
$$

with $\frac{1}{(1-\beta)\gamma} + \frac{1}{2(1-\beta)} = 1$. By Lemma 5.52, we obtain

$$
|(u_t, u)|^{\frac{1}{1-\beta}} \leq C\left(Z(t) + \|u\|_p^p + \|u_t\|^2 + k \circ \nabla_{\mathbb{G}} u\right). \tag{5.272}
$$

From this fact, we calculate

$$
\begin{aligned}
A(t) &= \left(Z^{1-\beta}(t)+\varepsilon(u_t,u)\right)^{\frac{1}{1-\beta}} \le 2^{\frac{1}{1-\beta}}\left(Z(t)+|(u_t,u)|^{\frac{1}{1-\beta}}\right)\\
&\le 2^{\frac{1}{1-\beta}}\left(Z(t)+C\left(Z(t)+\|u\|_p^p+\|u_t\|^2+k\circ\nabla_{\mathbb{G}}u\right)\right)\\
&\le C\left(Z(t)+\|u\|_p^p+\|u_t\|^2+k\circ\nabla_{\mathbb{G}}u\right)\\
&\le CA'(t), \quad \forall t\in[0,T].
\end{aligned}
\tag{5.273}
$$

Hence,

$$
A^{\frac{\beta}{1-\beta}}(t) \ge \frac{C(1-\beta)}{C(1-\beta)A^{-\frac{\beta}{1-\beta}}(0)-t\beta}, \tag{5.274}
$$

therefore, we arrive at

$$
T_B \le \frac{C(1-\beta)}{\beta(A(0))^{\frac{\beta}{1-\beta}}}. \tag{5.275}
$$

Therefore, $A(t)$ blows up in finite time. That is,

$$
\lim_{t\to T_B}\|\nabla_{\mathbb{G}}u\| = +\infty. \tag{5.276}
$$

$\square$

5.9. **Kato type exponents for the wave Rockland equations.** In one of the most popular works of Kato he considered the following problem:

$$
\frac{\partial^2 u(x,t)}{\partial t^2}-\Delta u(x,t)=|u(x,t)|^p, \quad (x,t):=\mathbb{R}^N\times(0,+\infty), \tag{5.277}
$$

for $N>1$ and $p>1$, with the Cauchy data

$$
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\mathbb{R}^N.
$$

For the wave problem (5.277), Kato's result states that if u is a generalised solution of the problem (5.277) with $u_0,u_1\in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp} u\subset\{|x|\le R+t\}$ and

$$
\int_{\mathbb{R}^N}|x|^{\eta-1}u_0(x)dx>0, \quad \int_{\mathbb{R}^N}u_1(x)dx>0,
$$

where

$$
\eta(N)=\begin{cases} 0 & \text{if } N \text{ is odd},\\ \frac{1}{2} & \text{if } N \text{ is even},\end{cases}
$$

then the solution u cannot be globally (in time) defined if

$$
1<p\le\frac{N+1}{N-1}. \tag{5.278}
$$

The exponent $p^*=\frac{N+1}{N-1}$ is usually called the Kato critical exponent for the problem (5.277).

The wave equation on the Heisenberg group $\mathbb{H}^n$ studied in [101], where the authors concerned the following problem

$$
\frac{\partial^2 u(x,t)}{\partial t^2}-\Delta_{\mathbb{H}^n}u(x,t)=|u(x,t)|^p, \quad (x,t):=\mathbb{H}^n\times(0,+\infty),
$$

with the Cauchy data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $p > 1$. Also, in the paper [102] it is considered a space–fractional analogue of the non-linear wave equation on the Heisenberg group

$$\frac{\partial^2 u(x,t)}{\partial t^2} + (-\Delta_{\mathbb{H}^n})^s |u(x,t)|^m = |u(x,t)|^p, \quad (x,t) := \mathbb{H}^n \times (0,+\infty),$$

with the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where $(-\Delta_{\mathbb{H}^n})^s$ is the fractional sub-Laplacian on $\mathbb{H}^n$, $s \in (0,2)$, $m \neq 1$, $p > 1$.

In this dissertation we are not only interested in studying of the wave equations, also, the pseudo-hyperbolic equations and systems on graded Lie groups are in the field of our interest. In particular, we extend nonexistence results obtained by Véron and Pohozaev [101] for the hyperbolic equation and by Kirane and Ragoub [103] for the pseudo-hyperbolic equation and system on the Heisenberg group to the case of the graded Lie groups.

5.9.1. *Wave equation case.* Assume that $m > 0$ and let consider the Cauchy problem for the nonlinear Rockland wave equation

$$\begin{cases} u_{tt}(x,t) + \mathcal{R}|u(x,t)|^m = |u(x,t)|^p, & (x,t) \in \mathbb{G} \times (0,+\infty) := \Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{G}, \\ u_t(x,0) = u_1(x), & x \in \mathbb{G}. \end{cases} \tag{5.279}$$

where $\mathcal{R}$ is the Rockland operator in the following form:

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}},$$

where $\nu_j \in \mathbb{N}$, $c_j \in \mathbb{R}$, $j = 1,\dots,n$, and ν_0 is any common multiple of $\nu_1,\dots,\nu_n$, ([3, Lemma 4.1.8]). By $c_j \in \mathbb{R}$, $j = 1,\dots,n$, we can choose $\mathcal{R}$ such that it will be positive. Also, we introduce this operator in the Section 2.2.

Let us give definition of the weak solution of the Rockland wave equation (5.279).

Definition 5.55. Assume that $u_1, u_0 \in L^1_{loc}(\mathbb{G})$. We say that the function $u \in L^{\max\{m,p\}}_{loc}(\Omega_T)$ $(\Omega_T = \mathbb{G} \times (0,T))$ is a local weak solution of (5.279) if the identity

$$\begin{aligned} &\int_{\Omega_T} u(x,t) \frac{\partial^2 \varphi(x,t)}{\partial t^2} dx dt + \int_{\Omega_T} |u(x,t)|^m \mathcal{R}\varphi(x,t) dx dt \\ &= -\int_{\mathbb{G}} u_0(x) \frac{\partial \varphi(x,0)}{\partial t} dx + \int_{\mathbb{G}} u_1(x)\varphi(x,0) dx + \int_{\Omega_T} |u(x,t)|^p \varphi(x,t) dx dt, \end{aligned} \tag{5.280}$$

holds for all test functions

$$\varphi \in C^2((0,T]; L^2(\mathbb{G})) \cap C([0,T]; H^\gamma(\mathbb{G})),$$

such that $\gamma = 2 \max\limits_{j=1,\dots,n} \frac{\nu_0}{\nu_j}$, $\varphi(x,T) = 0$ and $\varphi \geq 0$. If $T = +\infty$ then u is called a global weak solution.

Here, the space $H^{\gamma}(\mathbb{G})$ is the homogeneous Sobolev space related to the Rockland operator $\mathcal{R}$, for more details, see [3, 104] and Section 2.2.

Theorem 5.56. *Assume that $\mathbb{G}$ be a graded Lie group with homogeneous dimension $Q \geq 2$ and $\mu = \max\limits_{j=1,\dots,n} \frac{\nu_j}{\nu_0}$ be such that $\mu Q > 1$. Assume that $p > 1$, and $\int_{\mathbb{G}} u_1(x)dx \geq 0$. Then if*

$$1 \leq m < p < p_c = \frac{\mu Q m + 1}{\mu Q - 1}, \tag{5.281}$$

the Cauchy problem (5.279) admits no non-negative global weak solution other than trivial.

Proof. We prove this theorem by contradiction. Suppose that there exists a weak solution u for some $T > 0$. By using (5.279) and Definition 5.55, we get

$$\begin{aligned} -\int_{\mathbb{G}} u_0(x)\frac{\partial\varphi(x,0)}{\partial t}dx + \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u(x,t)|^p\varphi(x,t)dxdt \\ = \int_{\Omega_T} \frac{\partial^2\varphi(x,t)}{\partial t^2}u(x,t)dxdt + \int_{\Omega_T} |u(x,t)|^m\mathcal{R}\varphi(x,t)dxdt. \end{aligned} \tag{5.282}$$

By choosing $\varphi(x,t)$ such that

$$\frac{\partial\varphi}{\partial t}(x,0) = 0.$$

By using ε-Young's inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}, \;\; \frac{1}{p} + \frac{1}{p'} = 1, \;\; a, b \geq 0,$$

we obtain

$$\begin{aligned} \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx &+ \int_{\Omega_T} |u(x,t)|^p\varphi(x,t)dxdt \\ &= \int_{\Omega_T} \frac{\partial^2\varphi(x,t)}{\partial t^2}u(x,t)dxdt + \int_{\Omega_T} |u(x,t)|^m\mathcal{R}\varphi(x,t)dxdt \\ &\leq \frac{1}{4}\int_{\Omega_T} |u|^p\varphi dxdt + C\int_{\Omega_T} \varphi^{-\frac{1}{p-1}}\left|\frac{\partial^2\varphi}{\partial t^2}\right|^{\frac{p}{p-1}} dxdt \\ &\quad + \frac{1}{4}\int_{\Omega_T} |u|^p\varphi dxdt + C\int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt. \end{aligned} \tag{5.283}$$

Then, we have

$$\begin{aligned} \int_{\Omega_T} |u|^p\varphi dxdt &\leq \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u|^p\varphi dxdt \\ &\leq C\int_{\Omega_T} \varphi^{-\frac{1}{p-1}}\left|\frac{\partial^2\varphi}{\partial t^2}\right|^{\frac{p}{p-1}} dxdt + C\int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt. \end{aligned} \tag{5.284}$$

Assume that $\Phi : \mathbb{R}_+ \to [0,1]$ be a smooth nonincreasing function such that

$$\Phi(z) := \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 2. \end{cases} \tag{5.285}$$

For $R>0$, we define

$$\varphi(x,t)=\Phi\left(\frac{|x|^2}{R^\mu}\right)\Phi\left(\frac{t^2}{R^{2\frac{p-1}{p-m}}}\right),$$

where $\Phi\in C^\infty[0,+\infty)$. By Denoting the following vector fields acting to the variable $X_j={}_xX_j$. By denoting $\Omega_1:=\{x\in\mathbb{G}:0\le|\widetilde{x}|\le 2\}$ and $\Omega_2:=\{t:0\le\widetilde{t}\le 2\}$. By substituting $x=R^\mu\widetilde{x}$ and $t=R^{\frac{p-1}{p-m}}\widetilde{t}$ and from Proposition 2.4, we get

$$\begin{aligned}
&\int_{\Omega_T}|\mathcal{R}\varphi(x,t)|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=\int_{\Omega_T}\sum_{j=1}^{n}\left|(-1)^{\frac{\nu_0}{\nu_j}}\,{}_xX_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=\int_{\Omega_T}\sum_{j=1}^{n}R^{-2\frac{\nu_0 p}{\nu_j(p-m)}\mu}\left|(-1)^{\frac{\nu_0}{\nu_j}}\,{}_{\widetilde{x}}X_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&\overset{\mu=\max_{j=1,\dots,n}\frac{\nu_j}{\nu_0}}{\le}R^{-\frac{2p}{p-m}}\int_{\Omega_T}\sum_{j=1}^{n}\left|{}_{\widetilde{x}}X_j^{2\frac{\nu_0}{\nu_j}}\varphi(x,t)\right|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}dxdt\\
&=R^{\frac{-2p}{p-m}}R^{\mu Q}R^{\frac{p-1}{p-m}}\int_{\Omega_{TR^{-\frac{p-1}{p-m}}}}|\mathcal{R}_{\widetilde{x}}\varphi(R^\mu\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})|^{\frac{p}{p-m}}\varphi^{-\frac{m}{p-m}}d\widetilde{x}d\widetilde{t}\\
&\le CR^{\mu Q-\frac{p+1}{p-m}},
\end{aligned}\tag{5.286}$$

and also,

$$\begin{aligned}
&\int_{\Omega_T}\varphi^{-\frac{p+1}{p-1}}(x,t)\left|\varphi(x,t)\frac{\partial^2\varphi(x,t)}{\partial t^2}\right|^{p'}dxdt\\
&=\int_{\Omega_T}\varphi^{-\frac{p+1}{p-1}}(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\left|\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\frac{\partial^2\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})}{\partial\widetilde{t}^2}\right|^{p'}R^{\frac{-2p'(p-1)}{p-m}}dxdt\\
&\le R^{\frac{-2p'(p-1)}{p-m}+\mu Q+\frac{p-1}{p-m}}\int_{\Omega_{TR^{-\frac{p-1}{p-m}}}}\varphi^{-\frac{p+1}{p-1}}(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\\
&\times\left|\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})\frac{\partial^2\varphi(R\widetilde{x},R^{\frac{p-1}{p-m}}\widetilde{t})}{\partial\widetilde{t}^2}\right|^{p'}d\widetilde{x}d\widetilde{t}\\
&\le CR^{\mu Q-\frac{p+1}{p-m}}.
\end{aligned}\tag{5.287}$$

Hence, we have

$$\int_{\Omega_T}|u|^p\varphi dxdt\le CR^{\mu Q-\frac{p+1}{p-m}}.\tag{5.288}$$

If $1<m<p<p_c=\frac{\mu Qm+1}{\mu Q-1}$ with $\mu Q-1>0$ and $R\to\infty$, we get

$$\int_{\Omega_T}|u|^p\varphi dxdt\le 0.\tag{5.289}$$

Therefore, we get $u = 0$. That is a contradiction, completing the proof. □

Corollary 5.57. *In the Abelian case* $(\mathbb{R}^n, +)$ *with* $Q = n$, $\mathcal{R} = -\Delta$, *and by taking Euclidean distance instead of the quasi-norm, we claim the well-known results by Kato* [105].

Corollary 5.58. *By Lemma 4.1.7 in* [3], *if* $\mathbb{G}$ *is a stratified Lie groups with* $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, *where* $\Delta_{\mathbb{G}}$ *is a sub-Laplacian (i.e.,* $\nu_0 = \nu_1 = \ldots = \nu_n$, *then* $\mu = 1$*), we obtain Kato's exponent for the wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.59. *That is a well-known one of the particular case of the stratified Lie groups is the Heisenberg group (*[3, p.174]*). So, then in the case of the Heisenberg group with* $m = 1$ *and* $m \in \mathbb{N}$ *we obtain the results by* [101] *and* [102], *respectively.*

5.9.2. *Wave equation with linear damping term.* In this section we consider the initial problem for the wave equation on the graded Lie group

$$\begin{cases} u_{tt}(x,t) + \mathcal{R}|u(x,t)|^m + u_t(x,t) = |u(x,t)|^p, \; (x,t) \in \mathbb{G} \times (0,+\infty) := \Omega, \\ u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{G}, \\ u_t(x,0) = u_1(x), \quad x \in \mathbb{G}, \end{cases} \tag{5.290}$$

where $\mathcal{R}$ is the Rockland operator in the form

$$\mathcal{R} = \sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}},$$

and $m, p > 0$.

Definition 5.60. Suppose that $u_1, u_0 \in L^1_{loc}(\mathbb{G})$. We call that $u \in L^{\max\{m,p\}}_{loc}(\Omega_T)$ ($\Omega_T = \mathbb{G} \times (0,T)$) is a local weak solution of the equation (5.290) if the identity

$$\begin{aligned} \int_{\Omega_T} u(x,t) \frac{\partial^2 \varphi(x,t)}{\partial t^2} dxdt &+ \int_{\Omega_T} |u(x,t)|^m \mathcal{R}\varphi(x,t) dxdt \\ &- \int_{\Omega_T} u(x,t) \frac{\partial \varphi(x,t)}{\partial t} dxdt \\ &= \int_{\mathbb{G}} u_0(x) \left(\varphi(x,0) + \frac{\partial \varphi(x,0)}{\partial t} \right) dx \\ &+ \int_{\mathbb{G}} u_1(x) \varphi(x,0) dx \\ &+ \int_{\Omega_T} |u(x,t)|^p \varphi(x,t) dxdt, \end{aligned} \tag{5.291}$$

holds for all nonnegative test functions

$$\varphi \in C^2((0,T]; L^2(\mathbb{G})) \cap C^1([0,T]; H^\gamma(\mathbb{G})),$$

such that $\varphi(x,T) = 0$. In the case $T = +\infty$, the solution of the equation (5.291) u is called a global weak solution.

Theorem 5.61. *Let $\mathbb{G}$ be graded groups with the homogeneous dimension $Q \geq 2$ and $\mu = \max\limits_{j=1,\ldots,n} \frac{\nu_j}{\nu_0}$. Assume that $p > 1$ and $\int_{\mathbb{G}} u_1(x)dx \geq 0$. If*

$$1 < m < p < p_c = m + \frac{2}{\mu Q}, \tag{5.292}$$

then the Cauchy problem (5.279) admits no global weak nonnegative solution other than trivial.

Proof. Similarly to the previous theorem, we have

$$\begin{aligned}\int_{\Omega_T} |u|^p \varphi dxdt &\leq \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx + \int_{\Omega_T} |u|^p \varphi dxdt \\ &\leq C\int_{\Omega_T} \varphi^{-\frac{p+1}{p-1}} \left|\varphi \frac{\partial^2 \varphi}{\partial t^2}\right|^{p'} dxdt \\ &+ C\int_{\Omega_T} |\mathcal{R}\varphi|^{\frac{p}{p-m}} \varphi^{-\frac{m}{p-m}} dxdt \\ &+ C\int_{\Omega_T} \left|\frac{\partial \varphi}{\partial t}\right|^{p'} \varphi^{-\frac{1}{p-1}} dxdt.\end{aligned}$$

By assuming

$$\int_{\Omega_T} \left|\frac{\partial \varphi}{\partial t}\right|^{p'} \varphi^{-\frac{1}{p-1}} dxdt < \infty,$$

we obtain

$$\int_{\Omega_T} |u|^p \varphi dxdt \leq CR^{\mu Q - \frac{2}{p-m}}. \tag{5.293}$$

By substituting $x = R^{\mu}\widetilde{x}$ and $t = R^{\frac{2(p-1)}{p-m}}\widetilde{t}$ with $1 < m < p < p_c = m + \frac{2}{\mu Q}$ and, letting $R \to \infty$, we get

$$\int_{\Omega_T} |u|^p \varphi dxdt \leq 0. \tag{5.294}$$

Hence, we get $u = 0$. □

Corollary 5.62. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0 = \nu_1 = \ldots = \nu_n$, then $\mu = 1$), we obtain Kato's type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.63. *In the case of the Heisenberg group we obtain the result by [102].*

5.9.3. *Pseudo-hyperbolic equation case.* In this subsection we show blow-up result for the pseudo-hyperbolic equation with Rockland operator on graded Lie groups in the following form:

$$\begin{cases} u_{tt} + \mathcal{R}u_{tt} + \mathcal{R}u = |u|^p, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ p > 1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ \ x \in \mathbb{G}. \end{cases} \tag{5.295}$$

Let us give definition of the weak solution of the (5.295).

Definition 5.64. We say that u is a local weak solution to (5.295) on Ω with initial data $u(x,0) = u_0(x) \in L^1_{loc}(\mathbb{G})$, if $u \in L^p_{loc}(\Omega_T)$ and satisfies

$$\begin{aligned} \int_{\Omega_T} |u|^p \varphi dxdt + \int_{\mathbb{G}} u_1(x)(\varphi(x,0) + \mathcal{R}\varphi(x,0))dx \\ - \int_{\mathbb{G}} u_0(x)(\varphi_t(x,0) + \mathcal{R}\varphi_t(x,0))dx \\ = \int_{\Omega_T} u\varphi_{tt}dxdt + \int_{\Omega_T} u\mathcal{R}\varphi_{tt}dxdt + \int_{\Omega_T} u\mathcal{R}\varphi dx, dt \end{aligned}$$

for any test function φ with $\varphi(x,T) = \varphi_t(x,T) = 0$. The solution u is said global if it exists on $(0;\infty)$.

Theorem 5.65. *Let $\mathbb{G}$ be a graded Lie group with homogeneous dimension Q and $\mu = \max\limits_{j=1,\dots,n} \frac{\nu_j}{\nu_0}$ be such that $\mu Q > 1$. Assume that $u_1 \in L^1(\mathbb{G})$ and*

$$\int_{\mathbb{G}} u_1 dx > 0. \tag{5.296}$$

If

$$1 < p \leq p_c = 1 + \frac{2}{\mu Q - 1}, \tag{5.297}$$

then there exists no nontrivial global weak solution of (5.295).

Proof. Firstly, we consider the case $1 < p < p_c$. From Definition 5.64 with $\varphi_t(x,0) = 0$, we obtain

$$\begin{aligned} \int_{\Omega} |u|^p \varphi dxdt + \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx &= \int_{\Omega} u\varphi_{tt}dxdt \\ &+ \int_{\Omega} \mathcal{R}u\varphi_{tt}dxdt + \int_{\Omega} u\mathcal{R}\varphi dxdt - \int_{\mathbb{G}} u_1(x)\mathcal{R}\varphi(x,0)dx \\ &\leq \Big| \int_{\Omega} u\varphi_{tt}dxdt + \int_{\Omega} u\mathcal{R}\varphi_{tt}dxdt \\ &+ \int_{\Omega} u\mathcal{R}\varphi dxdt - \int_{\mathbb{G}} u_1(x)\mathcal{R}\varphi(x,0)dx \Big| \\ &\leq \int_{\Omega} |u\varphi_{tt}|dxdt + \int_{\Omega} |u\mathcal{R}\varphi_{tt}|dxdt \\ &+ \int_{\Omega} |u\mathcal{R}\varphi|dxdt + \int_{\mathbb{G}} |u_1(x)\mathcal{R}\varphi(x,0)|dx. \end{aligned} \tag{5.298}$$

Then from the Young inequality, we have

$$\begin{aligned} \int_{\Omega} |u||\varphi_{tt}|dxdt &= \int_{\Omega} |u|\varphi^{\frac{1}{p}}\varphi^{-\frac{1}{p}}|\varphi_{tt}|dxdt \\ &\leq \varepsilon \int_{\Omega} |u|^p \varphi dxdt + c_\varepsilon \int_{\Omega} \varphi^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt, \end{aligned}$$

$$\int_{\Omega}|u||\mathcal{R}\varphi_{tt}|dtdx \le \varepsilon\int_{\Omega}|u|^p\varphi dtdx + c_{\varepsilon}\int_{\Omega}\varphi^{-\frac{1}{p-1}}|\mathcal{R}\varphi_{tt}|^{\frac{p}{p-1}}dxdt, \tag{5.299}$$

and

$$\int_{\Omega}|u||\mathcal{R}\varphi|dtdx \le \varepsilon\int_{\Omega}|u|^p\varphi dtdx + c_{\varepsilon}\int_{\Omega}\varphi^{-\frac{1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dxdt, \tag{5.300}$$

for some positive constant c_{ε}. From using above facts, we get

$$\begin{aligned}\int_{\Omega}|u|^p\varphi dxdt &+ \int_{\mathbb{G}}u_1(x)\varphi(x,0)dx \\ &\le C\left(A_p(\varphi)+B_p(\varphi)+C_p(\varphi)+\int_{\mathbb{G}}|u_1(x)||\mathcal{R}\varphi(x,0)|dx\right),\end{aligned} \tag{5.301}$$

where

$$A_p(\varphi) = \int_{\Omega}\varphi^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt, \tag{5.302}$$

$$B_p(\varphi) = \int_{\Omega}\varphi^{-\frac{1}{p-1}}|\mathcal{R}\varphi_{tt}|^{\frac{p}{p-1}}dxdt, \tag{5.303}$$

$$C_p(\varphi) = \int_{\Omega}\varphi^{-\frac{1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dxdt. \tag{5.304}$$

Let us choose the following test function

$$\varphi_R(x,t) = \Phi\left(\frac{|x|^2}{R^{\mu}}\right)\Phi\left(\frac{t^2}{R^2}\right), \; R>0, \tag{5.305}$$

with the following property

$$\Phi(r) = \begin{cases}1, & \text{if } 0\le r<1,\\ \searrow, & \text{if } 1\le r<2,\\ 0, & \text{if } r\ge 2,\end{cases}$$

where $\Phi:\mathbb{R}_+\to[0,1]$ is a sufficiently smooth nonincreasing function. We note that

$$\frac{\partial\varphi_R(x,t)}{\partial t} = \frac{2t}{R^2}\Phi\left(\frac{|x|^2}{R^{\mu}}\right)\Phi_t'\left(\frac{t^2}{R^2}\right), \tag{5.306}$$

we have

$$\frac{\partial\varphi_R(x,0)}{\partial t} = 0. \tag{5.307}$$

Let us estimate $A_p(\varphi_R), B_p(\varphi_R), C_p(\varphi_R)$. By choosing variables $x = R^{\mu}\widetilde{x}$ and $t = R\widetilde{t}$, then

$$\widetilde{\Omega} := \{\overline{x}\in\mathbb{G} : 0\le|x|\le 2\} \text{ and } \widehat{\Omega} := \{\overline{t} : 0\le t^2\le 2\}. \tag{5.308}$$

By using Proposition 2.4, we calculate

$$A_p(\varphi_R) = \int_0^{\infty}\int_{\mathbb{G}}|\varphi_R(x,t)|^{-\frac{p}{p-1}}\left|\frac{\partial^2\varphi_R(x,t)}{\partial t^2}\right|^{\frac{p}{p-1}}dxdt \le CR^{\mu Q+1-\frac{2p}{p-1}}, \tag{5.309}$$

$$B_p(\varphi_R) \le CR^{\mu Q+1-\frac{4p}{p-1}}, \tag{5.310}$$

and

$$C_p(\varphi_R) \le CR^{\mu Q+1-\frac{2p}{p-1}}. \tag{5.311}$$

Also, we get that

$$|\mathcal{R}\varphi_R(x,t)| \leq CR^{-2}. \tag{5.312}$$

By combining the estimates (5.309)-(5.311) in (5.301), we get

$$\begin{aligned}\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ \leq C\Big(R^{\mu Q+1-\frac{2p}{p-1}} + R^{\mu Q+1-\frac{4p}{p-1}} + R^{\mu Q+1-\frac{4p}{p-1}} \\ + \int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\Big) \\ \leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + \int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\right).\end{aligned} \tag{5.313}$$

On the other hand, we get

$$\begin{aligned}\liminf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ \geq \lim\inf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt \\ + \lim\inf_{R\to\infty} \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx.\end{aligned}$$

By using the monotone convergence theorem, we obtain

$$\lim\inf_{R\to\infty} \int_{\Omega} |u|^p \varphi_R dxdt = \int_{\Omega} |u|^p dxdt.$$

Since $u_1 \in L^1(\mathbb{G})$, by the dominated convergence theorem, we have

$$\lim\inf_{R\to\infty} \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx = \int_{\mathbb{G}} u_1(x)dx.$$

Now, we have

$$\lim\inf_{R\to\infty} \left(\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx\right) \geq \int_{\Omega} |u|^p dxdt + d,$$

where

$$d = \int_{\mathbb{G}} u_1(x)dx > 0.$$

By the definition of the limit, for every $\varepsilon > 0$ exists $R_0 > 0$ such that

$$\begin{aligned}\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \\ > \lim\inf_{R\to\infty} (\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx) - \varepsilon \\ \geq \int_{\Omega} |u|^p dxdt + d - \varepsilon,\end{aligned}$$

for every $R \geq R_0$. By taking $\varepsilon = \frac{d}{2}$, we have

$$\int_{\Omega} |u|^p \varphi_R dxdt + \int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx \geq \int_{\Omega} |u|^p dxdt + \frac{d}{2},$$

for every $R \geq R_0$. Then from (5.313), (5.312) and $u_1 \in L^1(\mathbb{G})$, we have

$$\begin{aligned} \int_\Omega |u|^p dxdt + \frac{l}{2} &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + \int_{\widetilde{\Omega}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx\right) \\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + R^{-2}\int_{\widetilde{\Omega}} |u_1(x)|dx\right) \\ &\leq C\left(R^{\mu Q+1-\frac{2p}{p-1}} + R^{-2}\right). \end{aligned} \tag{5.314}$$

Thus, we obtain

$$\mu Q + 1 - \frac{2p}{p-1} < 0,$$

or

$$p < 1 + \frac{2}{\mu Q - 1}.$$

If $R \to \infty$, we get

$$\int_\Omega |u|^p dxdt + \frac{l}{2} \leq 0.$$

This is a contradiction. Finally, we obtain

$$\int_\Omega |u|^p dxdt + \frac{l}{2} = 0.$$

Let us consider another case $p = 1 + \frac{2}{\mu Q - 1}$. By using (5.314), we have

$$\int_\Omega |u|^p dxdt \leq C < \infty, \tag{5.315}$$

then

$$\lim_{R\to\infty} \int_\Omega |u|^p \varphi_R dxdt = 0. \tag{5.316}$$

Using the Hölder inequality instead of Young's inequality in (5.298), we get

$$\int_\Omega |u|^p \varphi_R dxdt + \frac{d}{2} \leq C\left(\int_{\widetilde{\Omega}} |u|^p \varphi_R dxdt\right)^{\frac{1}{p}}.$$

If $R \to \infty$ then by combining the above facts, we have

$$\int_\Omega |u|^p \varphi_R dxdt + \frac{d}{2} = 0.$$

This contradiction completes the proof. □

Corollary 5.66. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0 = \nu_1 = \ldots = \nu_n$, then $\mu = 1$) and $c_j = 1$, $j = 1, \ldots, n$, we obtain Kato-type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.67. *In the case of the Heisenberg group, in particular, we obtain the results of the paper* [103].

5.9.4. *The case of system.* Let us consider the system of the pseudo-hyperbolic Rockland equations with the Cauchy conditions:

$$\begin{cases} u_{tt} + \mathcal{R}u_{tt} + \mathcal{R}u = |v|^q, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ q > 1, \\ v_{tt} + \mathcal{R}v_{tt} + \mathcal{R}v = |u|^p, \ (x,t) \in \mathbb{G} \times (0,T) := \Omega_T, \ p > 1, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \ \ x \in \mathbb{G}, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), \ \ x \in \mathbb{G}. \end{cases} \tag{5.317}$$

Firstly, let us give a definition of the weak solution of (5.317) in the following form:

Definition 5.68. We say that the pair $(u; v)$ is a local weak solution of (5.317) on $\mathbb{G}$ with the Cauchy data $(u(x,0); v(x,0)) = (u_0; v_0) \in L^1_{loc}(\mathbb{G}) \times L^1_{loc}(\mathbb{G})$, if $(u,v) \in L^p_{loc}(\Omega_T) \times L^q_{loc}(\Omega_T)$ satisfies

$$\begin{aligned} \int_{\Omega_T} |v|^q \varphi dx dt &+ \int_{\mathbb{G}} u_1(x)(\varphi(x,0) + \mathcal{R}\varphi(x,0))dx \\ &- \int_{\mathbb{G}} u_0(x)(\varphi_t(x,0) + \mathcal{R}\varphi_t(x,0))dx \\ &= \int_{\Omega_T} u\varphi_{tt} dx dt + \int_{\Omega_T} u\mathcal{R}\varphi_{tt} dx dt + \int_{\Omega_T} u\mathcal{R}\varphi dx dt, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_T} |u|^p \varphi dx dt &+ \int_{\mathbb{G}} v_1(x)(\varphi(x,0) + \mathcal{R}\varphi(x,0))dx \\ &- \int_{\mathbb{G}} v_0(x)(\varphi_t(x,0) + \mathcal{R}\varphi_t(x,0))dx \\ &= \int_{\Omega_T} v\varphi_{tt} dx dt + \int_{\Omega_T} v\mathcal{R}\varphi_{tt} dx dt + \int_{\Omega_T} v\mathcal{R}\varphi dx dt, \end{aligned}$$

for any test function φ with $\varphi(\cdot, T) = \varphi_t(\cdot, T) = 0$. The solution is said to be a global if it exists for $T = +\infty$.

Now we present the main result in the system case.

Theorem 5.69. *Let $\mathbb{G}$ be a graded Lie group with homogeneous dimension Q and $\mu = \max\limits_{j=1,\dots,n} \frac{\nu_j}{\nu_0}$ be such that $\mu Q > 1$.Assume that $(u_1, v_1) \in L^1(\mathbb{G}) \times L^1(\mathbb{G})$ with*

$$\int_{\mathbb{G}} u_1 dx > 0, \ \ \textit{and} \ \ \int_{\mathbb{G}} v_1 dx > 0. \tag{5.318}$$

If $1 < pq \leq (pq)^* = 1 + \frac{2}{\mu Q - 1} \max\{p+1; q+1\}$ then there exists no nontrivial weak solution to (5.317).

Proof. Similarly, with the of single equation with $\varphi_t(x,0)=0$, we get

$$\begin{aligned}\int_\Omega |v|^q\varphi dxdt &+ \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx\\ &\leq \int_\Omega |u||\varphi_{tt}|dxdt + \int_\Omega |u||\mathcal{R}\varphi_{tt}|dxdt\\ &+ \int_\Omega |u||\mathcal{R}\varphi|dxdt + \int_{\mathbb{G}} |u_1(\vartheta)||\mathcal{R}\varphi(x,0)|dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt &+ \int_{\mathbb{G}} v_1(x)\varphi(x,0)dx\\ &\leq \int_\Omega |v||\varphi_{tt}|dxdt + \int_\Omega |v||\mathcal{R}\varphi_{tt}|dxdt\\ &+ \int_\Omega |v||\mathcal{R}\varphi|dxdt + \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi(x,0)|dx.\end{aligned}$$

By choosing $\varphi=\varphi_R$, the test function given by (5.305) and from the Hölder inequality, we calculate

$$\begin{aligned}\int_\Omega |v|^q\varphi_R dxdt &+ \int_{\mathbb{G}} u_1(x)\varphi(x,0)dx - \int_{\mathbb{G}} |u_1(x)||\mathcal{R}\varphi(x,0)|dx\\ &\leq (A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}})\left(\int_\Omega |u|^p\varphi_R dxdt\right)^{\frac{1}{p}},\end{aligned}$$

where $A_p(\varphi)$, $B_p(\varphi)$ and $C_p(\varphi)$ are given in the single equation case. Similarly, by the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi_R dxdt &+ \int_{\mathbb{G}} v_1(x)\varphi(x,0)dx - \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi(x,0)|dx\\ &\leq (A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}})\left(\int_\Omega |v|^q\varphi_R dxdt\right)^{\frac{1}{q}}.\end{aligned}$$

Assume that for the large R, we get

$$\begin{aligned}\int_{\mathbb{G}} u_1(x)\varphi_R(x,0)dx - \int_{\mathbb{G}} |u_1(x)||\mathcal{R}\varphi_R(x,0)|dx \geq 0,\\ \int_{\mathbb{G}} v_1(x)\varphi_R(x,0)dx - \int_{\mathbb{G}} |v_1(x)||\mathcal{R}\varphi_R(x,0)|dx \geq 0.\end{aligned} \tag{5.319}$$

Then, we have

$$\int_\Omega |v|^q\varphi_R dxdt \leq (A_p(\varphi_R)^{\frac{p-1}{p}} + B_p(\varphi_R)^{\frac{p-1}{p}} + C_p(\varphi_R)^{\frac{p-1}{p}})\left(\int_\Omega |u|^p\varphi_R dxdt\right)^{\frac{1}{p}}, \tag{5.320}$$

and

$$\int_\Omega |u|^p\varphi_R dxdt \leq (A_q(\varphi_R)^{\frac{q-1}{q}} + B_q(\varphi_R)^{\frac{q-1}{q}} + C_q(\varphi_R)^{\frac{q-1}{q}})\left(\int_\Omega |v|^q\varphi_R dxdt\right)^{\frac{1}{q}}. \tag{5.321}$$

By choosing variables $\overline{t} = R^{-1}t$ and $\overline{x} = R^{-\mu}x$, we get

$$\int_{\Omega} |v|^q \varphi_R dxdt \leq CR^{\frac{\mu Q(p-1)-(p+1)}{p}} \left(\int_{\Omega} |u|^p \varphi_R dxdt \right)^{\frac{1}{p}}, \tag{5.322}$$

and

$$\int_{\Omega} |u|^p \varphi_R dxdt \leq CR^{\frac{\mu Q(q-1)-(q+1)}{q}} \left(\int_{\Omega} |v|^q \varphi_R dxdt \right)^{\frac{1}{q}}. \tag{5.323}$$

By using the last two inequalities, we have

$$\left(\int_{\Omega} |u|^p \varphi_R dxdt \right)^{1-\frac{1}{pq}} \leq CR^{\alpha_1}, \tag{5.324}$$

and

$$\left(\int_{\Omega} |v|^q \varphi_R dxdt \right)^{1-\frac{1}{pq}} \leq CR^{\alpha_2}, \tag{5.325}$$

where

$$\alpha_1 = \frac{\mu Q(pq-1) - pq - 2q - 1}{pq}$$

and

$$\alpha_2 = \frac{\mu Q(pq-1) - pq - 2p - 1}{pq}.$$

Then we need $\alpha_1, \alpha_2 < 0$.

Secondly, let us consider the case $1 < pq < 1 + \frac{2}{\mu Q - 1} \max\{p+1; q+1\}$.

Case 1: $1 < pq < 1 + \frac{2}{\mu Q - 1} \max\{p+1; q+1\}$. By letting $R \to \infty$ in (5.324) with $1 < q \leq p$, we have

$$\int_{\Omega} |u|^p dxdt = 0,$$

which is a contradiction. Similarly, in the case $1 < p \leq q$, from (5.325), we have

$$\int_{\Omega} |v|^q dxdt = 0.$$

Case 2: $pq = 1 + \frac{2}{\mu Q - 1} \max\{p+1; q+1\}$. This case is similar with the proof of Theorem 5.65.

□

Corollary 5.70. *In the case $p = q$ and $u = v$ in Theorem 5.69, we arrive at a single equation given by Theorem 5.65.*

Proof. From Theorem 5.69, we get

$$p^2 \leq 1 + \frac{2(p+1)}{\mu Q - 1},$$

and

$$p^2 - 1 \leq \frac{2(p+1)}{\mu Q - 1}.$$

Then dividing both sides by $p+1$, we obtain

$$p-1 \leq \frac{2}{\mu Q-1}. \tag{5.326}$$

$\square$

Corollary 5.71. *In the case, if $\mathbb{G}$ is a stratified Lie groups with $\mathcal{R} = -\Delta_{\mathbb{G}} = -\sum_1^n X_i^2$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian (i.e., $\nu_0 = \nu_1 = \ldots = \nu_n$, then $\mu = 1$), we obtain Kato's type exponent for the linear damping wave equation with the sub-Laplacian on stratified Lie groups.*

Corollary 5.72. *In the case of the Heisenberg group, in particular, we obtain the result by* [103].

5.10. **Fujita type exponents for the heat Rockland equations.** In the one of the most popular works of Fujita in [106] considered the nonlinear heat equation

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = u^{1+p}, \ \ (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \ \ x \in \mathbb{R}^N, \end{cases} \tag{5.327}$$

for the subject of blowing up. He obtained that if $0 < p < \frac{2}{N}$ a solution of the problem (5.327) blows up in finite time for some $x_0 \in \mathbb{R}^N$, $N \in \mathbb{N}$. One of the further generalisations of the problem (5.327) is considering the fractional Laplacian $(-\Delta)^s$ instead of the classical Laplacian $-\Delta$. Namely, in [107, 108, 109] the authors considered the following Cauchy problem. In this dissertation we show Fujita's exponent for the heat Rockland operator and we show necessary condition for the global solvability.

The critical Fujita exponent determined as $p^* = 1 + \frac{2}{N}$ for the pseudo-parabolic equation in the Euclidean case were firstly established in the papers [110], [111]. In [112] authors studied the nonexistence of the global solutions to the nonlinear pseudo-parabolic equation on the Heisenberg group

$$u_t + (-\Delta_{\mathbb{H}^n})^m u_t + (-\Delta_{\mathbb{H}^n})^m u = |u|^p, (\eta,t) \in \mathbb{H}^n \times (0,\infty), \tag{5.328}$$

with the Cauchy data

$$u(\eta,0) = u_0(\eta), \eta \in \mathbb{H}^n, \tag{5.329}$$

where $m > 1, p > 1$, $\Delta_{\mathbb{H}^n}$ is the Kohn-Laplace operator on (2×2)-dimensional Heisenberg group $\mathbb{H}^n$. For more details, the reader referred to [112] and references therein, [113], [114].

5.10.1. *Fujita exponent for the Heat Rockland equation.* Let us consider the Cauchy problem for the nonlinear heat Rockland equation in the following form:

$$\begin{cases} u_t(x,t) + \mathcal{R}^\alpha \{u\}^m(x,t) = u^p(x,t), \ \ (x,t) \in \mathbb{G} \times (0,+\infty) := \Omega_\infty, \\ u(x,0) = u_0(x) \geq 0, \ \ \ x \in \mathbb{G}, \end{cases} \tag{5.330}$$

where $\alpha > 0$, $m \in \mathbb{N}$, and $\mathcal{R}$ is a Rockland operator of the k-th order, that is,

$$\mathcal{R} = \sum_{j=1}^n (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}}.$$

By $\mathcal{R}^\alpha$ we understand fractional Rockland operator as Proposition 2.14. Let us denote by $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ the space of test functions φ with a compact support $\operatorname{supp}\varphi \subset \Omega_T$

such that $\varphi, \partial_t\varphi$ and $\mathcal{R}^\alpha\varphi$ are continuous functions on Ω_T with compact supports $\operatorname{supp}\partial_t\varphi, \operatorname{supp}\mathcal{R}^\alpha\varphi \subset \Omega_T$, where $\Omega_T := \mathbb{G} \times (0, T)$ for some $T > 0$.

Let us give a definition of the weak solution to the equation (5.330).

Definition 5.73. Fix $T > 0$. Assume that $u_0 \in L^1(\Omega_T)$ ($\Omega_T = \mathbb{G} \times (0, T)$). Then we call the function $u \in L^p(\Omega_T) \cap L^m(\Omega_T)$ a local weak solution of (5.330) if the identity

$$\begin{aligned} -\int_{\Omega_T} u(x,t)\,\frac{\partial\varphi(x,t)}{\partial t}\,dxdt + \int_{\Omega_T} \{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)dxdt \\ = \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_{\Omega_T} u^p(x,t)\varphi(x,t)dxdt, \end{aligned} \tag{5.331}$$

holds for all positive test functions φ from $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ such that $\varphi(x, T) = 0$.

If it is allowed to be $T = +\infty$ then u is called a global weak solution of the equation (5.330).

Theorem 5.74. *Assume that $\mathbb{G}$ be the graded Lie group with homogeneous dimension $Q \geq 2$. Assume that*

$$1 < p \leq p_c = m + \frac{k\alpha}{Q}. \tag{5.332}$$

Then the Cauchy problem (5.330), admits no global weak nonnegative solutions other than trivial.

Proof. We prove this theorem by contradiction. By using (5.330) and Definition 5.55, we have

$$\begin{aligned} \int_{\Omega_T} |u|^p\varphi dxdt \leq \int_{\Omega_T} |u|^p\varphi dxdt + \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx \\ = -\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}dxdt + C\int_{\Omega_T} |u(x,t)|^m\mathcal{R}^\alpha\varphi(x,t)dxdt, \end{aligned} \tag{5.333}$$

for some constant $C > 0$.

From s-Young's inequality

$$ab \leq sa^l + \frac{1}{s}b^{l'},\ \ \frac{1}{l} + \frac{1}{l'} = 1,\ \ a, b \geq 0,$$

we get

$$\begin{aligned}\int_{\Omega_T} u^p\varphi dxdt &\le -\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}dxdt + C\int_{\Omega_T}|u(x,t)|^m\mathcal{R}^\alpha\varphi(x,t)dxdt\\ &= \int_{\Omega_T}\varphi^{-\frac{1}{p}}\frac{\partial\varphi(x,t)}{\partial t}(-u(x,t))\varphi^{\frac{1}{p}}dxdt + C\int_{\Omega_T}\varphi^{\frac{m}{p}}|u(x,t)|^m\mathcal{R}^\alpha\varphi(x,t)\varphi^{-\frac{m}{p}}dxdt\\ &\le \frac{1}{4}\int_{\Omega_T}u^p\varphi dxdt + C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + \frac{1}{4}\int_{\Omega_T}u^p\varphi dxdt\\ &\qquad + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt\\ &= \frac{1}{2}\int_{\Omega_T}u^p\varphi dxdt + C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt,\end{aligned} \tag{5.334}$$

where

$$C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt < \infty \tag{5.335}$$

and C, C_1, C_2 are positive constants, then

$$\int_{\Omega_T}u^p\varphi dxdt \le C_1\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt + C_2\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt. \tag{5.336}$$

Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to [0,1]$ be smooth nonincreasing functions such that

$$\Phi(z) := \begin{cases}1, & \text{if } 0 \le z \le 1,\\ 0, & \text{if } z \ge 2.\end{cases} \tag{5.337}$$

For $R > 0$, we define

$$\varphi(x,t) = \Phi\left(\frac{|x|}{R}\right)\Phi\left(\frac{t}{R^\beta}\right).$$

By substituting variables $x = R\widetilde{x}$ and $t = R^\beta\widetilde{t}$ and by using Proposition 2.4 and (5.335), we get

$$\int_{\Omega_T}\varphi^{-\frac{p'}{p}}(x,t)\left|\frac{\partial\varphi(x,t)}{\partial t}\right|^{\frac{p}{p-1}}dxdt \le CR^{-\beta\frac{p}{p-1}+Q+\beta}, \tag{5.338}$$

and

$$\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt \le CR^{-k\alpha\frac{p}{p-m}+Q+\beta}. \tag{5.339}$$

Then by from (5.338) and (5.339), we have

$$\int_{\Omega_T}|u|^p\varphi dxdt \le C(R^{-\beta\frac{p}{p-1}+Q+\beta} + R^{-k\alpha\frac{p}{p-m}+Q+\beta}). \tag{5.340}$$

Let us choose $\beta = Q(m-1) + k\alpha$. Then, if $1 < p < m + \frac{k\alpha}{Q}$, we obtain

$$\int_{\Omega_T}u^p dxdt = \lim_{R\to\infty}\int_{\Omega_T}u^p\varphi dxdt \le 0. \tag{5.341}$$

Hence, $u=0$. This is a contradiction. $\square$

5.10.2. *Necessary conditions for local and global existence.* In this subsection we present necessary conditions for the existence of local and global solutions to the equation (5.330).

Theorem 5.75. *Suppose $p>m$ and $\alpha>0$. Assume that u be a local solution to (5.330) for $T<\infty$. Then we have the estimate*

$$\lim_{|x|\to\infty} \inf u_0(x) \leq C\, T^{1-p'}, \tag{5.342}$$

for a positive constant C, where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. By denoting the following test function

$$\varphi(x,t)=\Phi\left(\frac{|x|}{R}\right)\Phi\left(\frac{t}{T}\right), \tag{5.343}$$

where Φ is a smooth nonnegative function with a compact support and

$$\Phi_2\left(\frac{t}{T}\right):=\begin{cases}\left(1-\frac{t}{T}\right)^l, & 0<t\leq T,\\ 0, & t>T,\end{cases} \tag{5.344}$$

where $l>p'-1$. By combining Definition 5.73 and s-Young's inequality, we get

$$\begin{aligned}
&\int_{\mathbb{G}} u_0(x)\varphi(x,0)dx+\int_{\Omega_T} u^p\varphi dxdt\\
&\quad\leq -\int_{\Omega_T}\frac{\partial\varphi(x,t)}{\partial t}u(x,t)dxdt+\int_{\Omega_T}\{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)dxdt\\
&\quad=\int_{\Omega_T}\varphi^{-\frac{1}{p}}\frac{\partial\varphi(x,t)}{\partial t}(-u(x,t))\varphi^{\frac{1}{p}}dxdt+\int_{\Omega_T}\varphi^{\frac{1}{p}}\{u\}^m(x,t)\mathcal{R}^\alpha\varphi(x,t)\varphi^{-\frac{1}{p}}dxdt\\
&\quad\leq\frac{1}{2}\int_{\Omega_T}u^p\varphi dxdt+C\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt+\frac{1}{2}\int_{\Omega_T}|u|^p\varphi dxdt\\
&\qquad\qquad+C\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt\\
&\quad=\int_{\Omega_T}u^p\varphi dxdt+C\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt+C\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt.
\end{aligned} \tag{5.345}$$

Finally, we have

$$\int_{\mathbb{G}} u_0(x)\varphi(x,0)dx\leq C\int_{\Omega_T}\varphi^{-\frac{p'}{p}}\left|\frac{\partial\varphi}{\partial t}\right|^{p'}dxdt+C\int_{\Omega_T}\varphi^{\frac{-m}{p-m}}|\mathcal{R}^\alpha\varphi|^{\frac{p}{p-m}}dxdt. \tag{5.346}$$

By substituting $t=T\widetilde{t}$ and $x=R\widetilde{x}$ and by using Proposition 2.4, we get

$$R^Q\int_{\mathbb{G}}u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x}\leq CR^QT^{\frac{-p'}{p}}\int_{\mathbb{G}}\Phi_1(\widetilde{x})d\widetilde{x}+CTR^{Q-\frac{\kappa\alpha p}{p-m}}\int_{\mathbb{G}}\Phi_1(\widetilde{x})d\widetilde{x}, \tag{5.347}$$

and, we obtain

$$\int_{\mathbb{G}} u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x} \leq CT^{\frac{-p'}{p}} \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x} + CTR^{-\frac{k\alpha p}{p-m}} \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}$$
$$= C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}. \quad (5.348)$$

Hence, we obtain

$$C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x} \geq \int_{\mathbb{G}} u_0(R\widetilde{x})\Phi_1(\widetilde{x})d\widetilde{x}$$
$$= \inf_{q(\widetilde{x})>1} (u_0(R\widetilde{x})) \int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}, \quad (5.349)$$

and by dividing to $\int_{\mathbb{G}} \Phi_1(\widetilde{x})d\widetilde{x}$ both sides, we get

$$\inf_{q(\widetilde{x})>1} u_0(R\widetilde{x}) \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}). \quad (5.350)$$

By letting $R \to \infty$, we have

$$\lim_{|x|\to\infty} \inf u_0(x) \leq CT^{\frac{-p'}{p}}. \quad (5.351)$$

□

Now, we show a necessary condition of the existence of the global solution.

Theorem 5.76. *Assume that $p > m$ and $\alpha > 0$ be such that $0 < \gamma < \frac{k\alpha}{p-m}$. Suppose that the problem (5.330) has a nontrivial and nonnegative global weak solution. Then the initial function u_0 satisfies the condition*

$$\lim_{|x|\to\infty} \inf(u_0(x)|x|^\gamma) \leq C, \quad (5.352)$$

where C is a positive constant independent of u.

Proof. Continuing discussions of the proof of the previous theorem, by (5.348), we have

$$\int_{\mathbb{G}} u_0(R\widetilde{x})\Phi(\widetilde{x})d\widetilde{x} \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} \Phi(\widetilde{x})d\widetilde{x}. \quad (5.353)$$

From $\operatorname{supp}\Phi \subset \{x : R < |x| < 2R\}$, we obtain

$$\begin{aligned}
&\inf_{|x|>R}(u_0(x)|x|^\gamma) \int_{\mathbb{G}} \Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x} \\
&\leq \int_{\mathbb{G}} u_0(R\widetilde{x})|R\widetilde{x}|^{p'-1}\Phi(\widetilde{x})|R\widetilde{x}|^{1-p'}d\widetilde{x} \\
&\leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}}) \int_{\mathbb{G}} |R\widetilde{x}|^\gamma \Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x} \\
&\leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}})R^\gamma \int_{\mathbb{G}} \Phi(\widetilde{x})|R\widetilde{x}|^{-\gamma}d\widetilde{x}.
\end{aligned} \quad (5.354)$$

Since $0 < \gamma < \frac{k\alpha}{p-m}$, we have

$$\inf_{|x|>R}(u_0(x)|x|^{\gamma}) \leq C(T^{\frac{-p'}{p}} + TR^{-\frac{k\alpha p}{p-m}})R^{\gamma}. \tag{5.355}$$

By changing $T = R^{\gamma(p-1)}$, we get

$$\inf_{|x|>R}(u_0(x)|x|^{\gamma}) \leq C(1 + R^{-(\frac{k\alpha}{p-m}-\gamma)p}), \tag{5.356}$$

and as $R \to \infty$, we have

$$\inf_{|x|\to\infty}(u_0(x)|x|^{\gamma}) \leq C. \tag{5.357}$$

$\square$

5.10.3. *Fujita exponent for the pseudo-parabolic Rockland equation.* In this subsection, we concern nonexistence of global weak solutions to the following nonlinear pseudo–parabolic equation

$$u_t(x,t)+\mathcal{R}u_t(x,t)+\mathcal{R}u(x,t) = |u(x,t)|^p+f(x,t), \quad (x,t) \in \mathbb{G}\times(0,\infty) := \Omega, \tag{5.358}$$

under the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{G}. \tag{5.359}$$

Similarly, with the heat Rockland equation case, We denote by $\mathfrak{C}^{\alpha,1}_{x,t}(\Omega_T)$ the space of test functions φ with a compact support $supp\,\varphi \subset \Omega_T$ such that $\varphi, \partial_t\varphi$, $\mathcal{R}\varphi$ and $\partial_t\mathcal{R}\varphi$ are continuous functions on Ω_T with compact supports $supp\,\partial_t\varphi$, $supp\,\mathcal{R}\varphi$, $supp\,\partial_t\mathcal{R}\varphi \subset \Omega_T$.

Definition 5.77. We say that u is a global weak solution to the problem (5.358)–(5.359) on Ω with the initial data $u(\cdot,0) = u_0(\cdot) \in L^1_{loc}(\mathbb{G})$, if $u \in L^p_{loc}(\Omega)$ and satisfies

$$\begin{aligned}&\int_\Omega |u|^p\varphi dxdt + \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt\\ &= -\int_\Omega u\varphi_t dxdt + \int_\Omega u(\mathcal{R}\varphi)_t dxdt - \int_\Omega u\mathcal{R}\varphi dxdt + \int_{\mathbb{G}} u_0(x)\mathcal{R}\varphi(x,0)dx\end{aligned} \tag{5.360}$$

for any regular test function φ with $\varphi(\cdot,t) = 0$ for large enough t.

For $R > 0$, we define

$$\Gamma_R = \{(x,t) \in \Omega : 0 \leq t \leq R^{\alpha},\ 0 \leq |x| \leq R\}.$$

Theorem 5.78. *Suppose that $\mathcal{R}$ is a Rockland operator of k-th order. Let $u_0 \in L^1(\mathbb{G})$ and $f^- \in L^1(\Omega)$, where $f^- = \max\{-f, 0\}$. Suppose that*

$$\int_{\mathbb{G}} u_0 dx + \lim_{R\to\infty}\inf\int_{\Gamma_R} f dxdt > 0. \tag{5.361}$$

If $1 < p \leq p^* = 1 + \frac{k}{Q}$, then the problem (5.358)–(5.359) does not admit any global weak solution.

Proof. Suppose that u is a global weak solution to the problem (5.358)–(5.359). Then, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt + \int_{\mathbb{G}} u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt \\ \leq \int_\Omega |u||\varphi_t|dxdt + \int_\Omega |u||(\mathcal{R}\varphi)_t|dxdt - \int_\Omega |u||\mathcal{R}\varphi|dxdt \\ + \int_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx.\end{aligned} \tag{5.362}$$

By using the ε-Young's inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{p'}, \ \ \frac{1}{p}+\frac{1}{p'}=1, \ \ a,b\geq 0,$$

with parameters p and $p/(p-1)$, we obtain

$$\int_\Omega |u||\varphi_t|dxdt \leq \varepsilon \int_\Omega |u|^p\varphi dxdt + c_\varepsilon \int_\Omega \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}dxdt, \tag{5.363}$$

for some positive constant c_ε.

Similarly, we have

$$\int_\Omega |u||(\mathcal{R}\varphi)_t|dxdt \leq \varepsilon \int_\Omega |u|^p\varphi dxdt + c_\varepsilon \int_\Omega \varphi^{\frac{-1}{p-1}}|(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}}dxdt, \tag{5.364}$$

and

$$\int_\Omega |u||\mathcal{R}\varphi|dxdt \leq \varepsilon \int_\Omega |u|^p\varphi dtdt + c_\varepsilon \int_\Omega \varphi^{\frac{-1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dtdt. \tag{5.365}$$

By using (5.362)-(5.365), for $\varepsilon>0$ small enough, we have

$$\begin{aligned}\int_\Omega |u|^p\varphi dxdt + \int_\Omega u_0(x)\varphi(x,0)dx + \int_\Omega f\varphi dxdt \\ \leq C\Big(A_p(\varphi)+B_p(\varphi)+C_p(\varphi)+\int_{\mathbb{G}} |u_0(x)||\mathcal{R}\varphi(x,0)|dx\Big),\end{aligned} \tag{5.366}$$

where

$$A_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|\varphi_t|^{\frac{p}{p-1}}dxdt, \tag{5.367}$$

$$B_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|(\mathcal{R}\varphi)_t|^{\frac{p}{p-1}}dxdt, \tag{5.368}$$

$$C_p(\varphi) = \int_\Omega \varphi^{\frac{-1}{p-1}}|\mathcal{R}\varphi|^{\frac{p}{p-1}}dxdt. \tag{5.369}$$

Let $\Phi_1, \Phi_2 : \mathbb{R}_+ \to [0,1]$ be smooth nonincreasing functions such that

$$\Phi_i(\rho) := \begin{cases} 1, & \text{if } 0\leq\rho\leq 1, \\ 0, & \text{if } \rho \geq 2, \end{cases} \tag{5.370}$$

for $i=1,2$.

Now, for $R>0$, let us consider the test function

$$\varphi_R(x,t) = \Phi_1\left(\frac{|x|}{R}\right)\Phi_2\left(\frac{t}{R^\alpha}\right),$$

for some $\alpha > 0$ to be defined later.

We observe that $supp\, \varphi_R$ is a subset of

$$\Omega_R = \{(x,t) \in \Omega : 0 \leq t \leq 2R^\alpha,\ 0 \leq |x| \leq 2R\},$$

while $supp\, \partial_t \varphi_R$, $supp\, \mathcal{R}\varphi_R$ and $supp\, \partial_t \mathcal{R}\varphi_R$ are subsets of

$$\Theta_R = \{(x,t) \in \Omega : R^\alpha \leq t \leq 2R^\alpha,\ R \leq |x| \leq 2R\},$$

also, we put

$$\Gamma_R = \{(x,t) \in \Omega : 0 \leq t \leq R^\alpha,\ 0 \leq |x| \leq R\}.$$

It follows that there is a positive constant $C > 0$, independent of R, such that for all $(x,t) \in \Omega_R$, we have

$$|\mathcal{R}_x \varphi_R(t,x)| \leq C R^{-k} \chi(t,x), \tag{5.371}$$

where $\chi(t,x)$ is a nonnegative function with a compact support in Ω_R, and

$$|\partial_t \mathcal{R} \varphi_R(t,x)| \leq C R^{-k-\alpha} \xi(t,x), \tag{5.372}$$

where $\xi(t,x)$ is a nonnegative function with a compact support in Ω_R.

Using (5.371) and (5.372), we get

$$A_p(\varphi) \leq C R^{\frac{-\alpha p}{p-1}}, \tag{5.373}$$

$$B_p(\varphi_R) \leq C R^{\frac{-(k+\alpha)p}{p-1}}, \tag{5.374}$$

$$C_p(\varphi_R) \leq C R^{\frac{-kp}{p-1}}. \tag{5.375}$$

Let us consider now the change of variables

$$\tilde{t} = R^{-\alpha} t, \quad \tilde{x} = R^{-1} x.$$

Put $\Sigma_R = \{x \in \mathbb{G} : R \leq |x| \leq 2R\}$.

By combining Proposition 2.4, (5.373), (5.374) and (5.375) in (5.366) we get

$$\begin{aligned} &\int_\Omega |u|^p \varphi_R dx dt + \int_\Omega u_0(x) \varphi_R(x,0) dx dt + \int_\Omega f \varphi_R dx dt \\ &\leq C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(v)||\mathcal{R}\varphi_R(0,v)| dv\Big), \end{aligned} \tag{5.376}$$

where

$$\lambda_1 = Q + \alpha - \frac{\alpha p}{p-1}$$

and

$$\lambda_2 = Q + \alpha - \frac{(k+\alpha)p}{p-1}$$

and

$$\lambda_3 = Q + \alpha - \frac{kp}{p-1}.$$

On the other hand, we have

$$\begin{aligned}&\liminf_{R\to\infty}\Big(\int_\Omega |u|^p\varphi_R dxdt+\int_{\mathbb{G}} u_0(x)\varphi_R(x,0)dx+\int_\Omega f\varphi_R dxdt\Big)\\ \geq &\liminf_{R\to\infty}\int_\Omega |u|^p\varphi_R dxdt+\liminf_{R\to\infty}\int_\Omega u_0(x)\varphi_R(x,0)dx+\liminf_{R\to\infty}\int_\Omega f\varphi_R dxdt.\end{aligned}$$

From the monotone convergence theorem, we get

$$\liminf_{R\to\infty}\int_\Omega |u|^p\varphi_R dxdt=\int_\Omega |u|^p dxdt.$$

Since $u_0\in L^1(\Omega)$, by the dominated convergence theorem, we have

$$\liminf_{R\to\infty}\int_{\mathbb{G}} u_0(x)\varphi_R(x,0)dx=\int_{\mathbb{G}} u_0(x)dx.$$

By denoting $f=f^+-f^-$, where $f^+=\max\{f,0\}$, we have

$$\begin{aligned}\int_\Omega f\varphi_R dxdt&=\int_{\Gamma_R} f dxdt+\int_{\Theta_R} f^+\varphi_R dxdt-\int_{\Theta_R} f^-\varphi_R dxdt\\ &\geq \int_{\Gamma_R} f dxdt-\int_{\Theta_R} f^-\varphi_R dxdt.\end{aligned}$$

Since $f^-\in L^1(\Omega)$, by the dominated convergence theorem we have

$$\lim_{R\to\infty}\int_{\Theta_R} f^-\varphi_R dxdt=0$$

Then

$$\liminf_{R\to\infty}\int_\Omega f\varphi_R dxdt\geq \liminf_{R\to\infty}\int_{\Gamma_R} f dxdt.$$

Then, we get

$$\begin{aligned}&\liminf_{R\to\infty}\Big(\int_\Omega |u|^p\varphi_R dxdt+\int_\Omega u_0(x)\varphi_R(x,0)dx+\int_\Omega f\varphi_R dxdt\Big)\\ &\geq \int_\Omega |u|^p dxdt+\ell.\end{aligned}$$

where form (5.361),

$$\ell=\int_\Omega u_0(x)dx+\liminf_{R\to\infty}\int_{\Gamma_R} f dxdt>0.$$

By the definition of the limit inferior, for every $\varepsilon>0$, there exists $R_0>0$ such that

$$\int_\Omega |u|^p\varphi_R dxdt+\int_\Omega u_0(x)\varphi_R(x,0)dx+\int_\Omega f\varphi_R dxdt$$

$$> \lim_{R\to\infty}\inf\Big(\int_\Omega |u|^p\varphi_R dxdt + \int_\Omega u_0(x)\varphi_R(x,0)dx + \int_\Omega f\varphi_R dxdt\Big) - \varepsilon$$

$$\geq \int_\Omega |u|^p dxdt + \ell - \varepsilon,$$

for every $R \geq R_0$. Taking $\varepsilon = \ell/2$, we get

$$\int_\Omega |u|^p\varphi_R dxdt + \int_\Omega u_0(x)\varphi_R(x,0)dx + \int_\Omega f\varphi_R dxdt$$

$$\geq \int_\Omega |u|^p dxdt + \frac{\ell}{2},$$

for every $R \geq R_0$. From (5.376), we have

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq C\Big(R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3} + \int_{\Sigma_R} |u_0(x)||\mathcal{R}_x\varphi_R(x,0)|dx\Big), \tag{5.377}$$

for R large enough.

Now, we take $\alpha = k$ and require that $\lambda = \max\{\lambda_1,\lambda_2,\lambda_3\} \leq 0$, which is equivalent to $1 < p \leq 1 + \frac{k}{Q}$. We consider two cases.

- Case 1. If $1 < p < 1 + \frac{k}{Q}$.

In this case, letting $R \to \infty$ in (5.377) and using the dominated convergence theorem, we obtain

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq 0,$$

which is a contradiction with $\ell > 0$.

- Case 2. If $p = 1 + \frac{k}{Q}$.

In this case, from (5.377), we obtain

$$\int_\Omega |u|^p dxdt \leq C < \infty. \tag{5.378}$$

By using the Hölder inequality with parameters p and $p/(p-1)$ and from (5.362), we get

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} \leq C\Big(\int_{\Theta_R} |u|^p\varphi_R dxdt\Big)^{\frac{1}{p}}.$$

By letting $R \to \infty$ in the above inequality and using (5.378), we have

$$\int_\Omega |u|^p dxdt + \frac{\ell}{2} = 0$$

This contradiction completes the proof of the theorem. $\square$

6. Appendix

In this appendix we deal with new inequalities related to the fractional order differential operators. Especially, the Caputo derivative analogues of the above inequalities are in the field of our interest. Here, we derive the generalizations of the classical Sobolev, Hardy, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities. Note that in this direction systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the book [4]. Also, we obtain these inequalities for Hadamard fractional derivative.

One of the Lyapunov's classical result in [67], he established that if $q \in C([a,b];\mathbb{R})$, for the boundary value problem

$$\begin{cases} u''(t) + q(t)u(t) = 0, \; x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{6.1}$$

has a nontrivial classical solution, then we have

$$\int\limits_a^b |q(s)|\, ds > \frac{4}{b-a}. \tag{6.2}$$

In [115], Hartman and Wintner generalised Lyapunov's inequality, it means if (6.1) has a nontrivial solution, then

$$\int\limits_a^b (b-s)(s-a)q^+(s)ds > b-a, \tag{6.3}$$

where $q^+(s) = \max\{q(s), 0\}$. Generalisation of the Lyapunov's inequality (6.2) can be obtained from (6.3) using the fact that $\max\limits_{a\le s\le b}(b-s)(s-a) = \frac{(b-a)^2}{4}$. Recently, some Hartman-Wintner-type inequalities were obtained for different fractional boundary value problems [116, 117].

In the [118], De La Vallée Poussin showed the following result:

Theorem 6.1. *Suppose that $u \in C^2([a,b])$ is a nontrivial solution to*

$$\begin{cases} -u''(x) - g(x)u'(x) = f(x)u(x), \; x \in (a,b), \\ u(a) = 0, \; u(b) = 0, \end{cases} \tag{6.4}$$

for $f, g \in C([a,b])$. Then

$$1 < M_1(b-a) + M_2\frac{(b-a)^2}{2}, \tag{6.5}$$

where $M_1 = \max\limits_{x\in[a,b]} |g(x)|$ and $M_2 = \max\limits_{x\in[a,b]} |f(x)|$.

As example, generalisation of the inequality (6.5) can be found in[119, 120]. Also, generalisation of above inequalities to the multidimensional case were generalised in the works [121, 122]. Motivated by the above cited works, using the approach introduced in [121, 122], some generalisations of above mentioned inequalities are established for fractional partial differential equations with Dirichlet conditions. Our results are natural generalizations of results in [122, 121]. In this dissertation, we established these inequalities for the fractional order derivatives.

Let us recall the Riemann–Liouville fractional integrals and derivatives. Also, we give definitions of the Caputo fractional derivatives. In ([123], p. 394) the sequential differentiation was formulated in a way that we will use in the further investigations. We refer to [124, 123] and references therein for further properties.

Definition 6.2. The left Riemann–Liouville fractional integral I_{a+}^{α} of order $\alpha > 0$, and right Riemann–Liouville I_{b-}^{α} of order $0 < \alpha \leq 1$ are given by

$$I_{a+}^{\alpha}\left[f\right](t) = \frac{1}{\Gamma(\alpha)}\int\limits_{a}^{t}(t-s)^{\alpha-1}f(s)ds, \;\; t \in (a,b],$$

and

$$I_{b-}^{\alpha}\left[f\right](t) = \frac{1}{\Gamma(\alpha)}\int\limits_{t}^{b}(s-t)^{\alpha-1}f(s)ds, \;\; t \in [a,b),$$

respectively. Here Γ denotes the Euler gamma function.

The left Riemann–Liouville fractional derivative D_{a+}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a continuous function f on $[a,b]$ is defined by

$$D_{a+}^{\alpha}\left[f\right](t) = \frac{d}{dt}I_{a+}^{1-\alpha}\left[f\right](t), \;\; \text{for any} \;\; t \in (a,b].$$

Similarly, the right Riemann–Liouville fractional derivative D_{b-}^{α} of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a continuous function f on $[a,b]$ is given by

$$D_{b-}^{\alpha}\left[f\right](t) = -\frac{d}{dt}I_{b-}^{1-\alpha}\left[f\right](t), \;\; \text{for any} \;\; t \in [a,b).$$

and

$$D_{a+}^{\alpha}\left[f\right](t) = \frac{d}{dt}I_{a+}^{1-\alpha}\left[f\right](t), \;\; t \in (a,b],$$

respectively and $f \in AC[a,b]$. Here Γ denotes the Euler gamma function.

Since $I^{\alpha}f(t) \to f(t)$ almost everywhere as $\alpha \to 0$, then by definition we suppose that $I^{0}f(t) = f(t)$. Hence $D_{a+}^{1}f(t) = f'(t)$.

Definition 6.3. The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a differentiable function f on $[a,b]$ are defined by

$$\mathcal{D}_{a+}^{\alpha}\left[f\right](t) = D_{a+}^{\alpha}\left[f(t) - f(a)\right], \;\; t \in (a,b],$$

and

$$\mathcal{D}_{b-}^{\alpha}\left[f\right](t) = D_{b-}^{\alpha}\left[f(t) - f(b)\right], \;\; t \in [a,b),$$

respectively.

Remark 6.4. *In Definition 6.3, if $f(a) = 0$, then $\mathcal{D}_{a+}^{\alpha} = D_{a+}^{\alpha}$.*

Proposition 6.5. *If $f \in L^{1}([a,b])$ and $\alpha > 0$, $\beta > 0$, then the following equality holds*

$$I_{a+}^{\alpha}I_{a+}^{\beta}f(t) = I_{a+}^{\alpha+\beta}f(t).$$

Proposition 6.6 ([123])**.** *If $f \in L^{1}([a,b])$ and $f' \in L^{1}([a,b])$, then the equality*

$$I_{a}^{\alpha}\mathcal{D}_{a+}^{\alpha}f(t) = f(t) - f(a), \; 0 < \alpha \leq 1,$$

holds almost everywhere on $[a,b]$.

Let us give some defintion of the Hadamard fractional derivative.

Definition 6.7. The left Hadamard fractional integrals $\mathfrak{I}^{\alpha}_{a+}$ of order $\alpha > 0$, and derivatives $\mathfrak{D}^{\alpha}_{a+}$ of order $0 < \alpha < 1$ are given by

$$\mathfrak{I}^{\alpha}_{a+}\left[f\right](t) = \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s\right)\frac{ds}{s},\;\; t \in (a,b],$$

and

$$\mathfrak{D}^{\alpha}_{a+}\left[f\right](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{-\alpha} f'\left(s\right)\frac{ds}{s}.\;\; t \in (a,b].$$

Here Γ denotes the Euler gamma function.

Proposition 6.8 ([123])**.** *If $f \in L^1(a,b)$ and $f' \in L^1_{\frac{1}{x}}(a,b)$, then the equality*

$$\mathfrak{I}^{\alpha}_{a}\mathfrak{D}^{\alpha}_{a+}f(t) = f(t) - f(a),\, 0 < \alpha < 1,$$

holds almost everywhere on $[a,b]$.

Then let us define weighted Lebesgue space with the norm:

$$\|u\|_{L^p_{\frac{1}{x}}(a,b)} := \left(\int_a^b |u(x)|^p \frac{dx}{x}\right)^{\frac{1}{p}}. \tag{6.6}$$

Proposition 6.9 ([123])**.** *If $f \in L^1_{\frac{1}{x}}(a,b)$ and $\alpha > 0$, $\beta > 0$, then the following equality holds*

$$\mathfrak{I}^{\alpha}_{a+}\mathfrak{I}^{\beta}_{a+}f(t) = \mathfrak{I}^{\alpha+\beta}_{a+}f(t).$$

Let us give definitions of the fractional and fractional p-Laplacian on $\mathbb{R}^N$:

Definition 6.10. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $0 < s < 1$. The fractional Laplacian operator of order s of a function $u \in C_0^\infty\left(\mathbb{R}^N\right)$ is defined by

$$(-\Delta)^s u(x) = 2\lim_{\delta\searrow 0}\int_{\mathbb{R}^N\setminus B(x,\delta)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,\;\; x \in \mathbb{R}^N, \tag{6.7}$$

where $B(x,\delta)$ is a ball at centered at $x \in \mathbb{R}^N$ with radius δ.

Definition 6.11. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $0 < s < 1$ and $1 < p < \infty$. The fractional p-Laplacian operator of order s of a function $u \in C_0^\infty\left(\mathbb{R}^N\right)$ is defined by

$$(-\Delta)^s_p u(x) = 2\lim_{\delta\searrow 0}\int_{\mathbb{R}^N\setminus B(x,\delta)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}dy,\;\; x \in \mathbb{R}^N. \tag{6.8}$$

In this appendix we derive the main results of the dissertation. In this subsection we show fractional order Poincaré–Sobolev type inequality.

6.1. **Poincaré–Sobolev type inequality for the Caputo fractional derivative.**

Theorem 6.12. *Let* $u \in L^p(a,b)$, $u(a) = 0$, $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ *and* $p > 1$. *Then for the* $\mathcal{D}^{\alpha}_{a+}$ *Caputo fractional derivative of order* $\alpha \in \left(\frac{1}{p}, 1\right]$ *we have the inequality*

$$\|u\|_{L^{\infty}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.9}$$

Proof. Let $u \in L^p(a,b)$, $u(a) = 0$, $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ and consider the function

$$u(t) = I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u(t). \tag{6.10}$$

Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned}
\left|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u(t)\right| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left|(t-s)^{\alpha-1}\mathcal{D}^{\alpha}_{a+}u(s)\right| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_a^t (t-s)^{\alpha q - q} ds\right)^{\frac{1}{q}} \left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\overset{\alpha > \frac{1}{p}}{=} \frac{(t-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\leq \frac{(b-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{(b-a)^{\alpha-\frac{1}{p}}}{(\alpha q - q + 1)^{\frac{1}{q}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)},
\end{aligned}$$

where $q = \frac{p}{p-1} > 1$.

Then,

$$\|u\|_{L^{\infty}(a,b)} = \|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}u\|_{L^{\infty}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.11}$$

showing (6.9). ∎

Remark 6.13. *In Theorem 6.12, by taking* $1 < q < \infty$, *we obtain*

$$\|u\|_{L^{q}(a,b)} \leq \frac{(b-a)^{\alpha-\frac{1}{p}+\frac{1}{q}}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}} \Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.12}$$

Theorem 6.14. *Also, Theorem 6.12 holds for the Riemann-Liouville derivative.*

Proof. By Theorem 6.12, we have $u(a) = 0$ and by using Remark 6.4, we have $\mathcal{D}^{\alpha}_{a+} = D^{\alpha}_{a+}$. $\square$

Let us also present the following result.

Theorem 6.15. *Let $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ with $p > 1$ and let $\beta \in [0,1)$ be such that $\alpha \in \left(\beta + \frac{1}{p}, 1\right]$. Then for the Caputo fractional derivative $\mathcal{D}^{\beta}_{a+}$, we have*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.13}$$

for all $1 < p \leq q < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Definition 6.3 and Properties 6.5, 6.6 we introduce the function

$$\mathcal{D}^{\beta}_{a+}u(t) = I^{1-\beta}_{a+}u'(t) = I^{\alpha-\beta}_{a+}I^{1-\alpha}_{a+}u'(t) = I^{\alpha-\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(t). \tag{6.14}$$

Using the Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
\left|I^{\alpha-\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(t)\right| &\leq \frac{1}{\Gamma(\alpha-\beta)}\int_a^t \left|(t-s)^{\alpha-\beta-1}\mathcal{D}^{\alpha}_{a+}u(s)\right| ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\left(\int_a^t (t-s)^{\alpha q-\beta q-q}ds\right)^{\frac{1}{q}}\left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&= \frac{(t-a)^{\alpha-\beta-1+\frac{1}{q}}}{(\alpha q-\beta q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)}\left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&= \frac{(t-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q-\beta q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)}\left(\int_a^t \left|\mathcal{D}^{\alpha}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}} \\
&\leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q-\beta q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)},
\end{aligned}$$

where by assumption $\alpha > \beta + \frac{1}{p}$, we have $\alpha q - \beta q - q + 1 > 0$. From this, we obtain

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}, \tag{6.15}$$

showing (6.13). $\square$

Remark 6.16. *In (6.13), if $\beta = 0$, we obtain Sobolev type inequality.*

Remark 6.17. *In Theorem 6.15, by taking $1 < q < \infty$, we get*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^q(a,b)} \leq \frac{(b-a)^{\alpha-\beta-\frac{1}{p}+\frac{1}{q}}}{(\alpha q - \beta q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha-\beta)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)}. \tag{6.16}$$

6.2. **Hardy type inequality for the Caputo fractional derivative.** Let us show Hardy inequality.

Theorem 6.18. *Let $a>0$, $u(a)=0$ and $\mathcal{D}_{a+}^{\alpha}u \in L^p(a,b)$ with $p>1$. Then for the $\mathcal{D}_{a+}^{\alpha}$ Caputo fractional derivative of order $\alpha \in \left(\frac{1}{p},1\right]$ we have the inequality*

$$\left\|\frac{u}{x}\right\|_{L^p(a,b)} \leq \frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathcal{D}_{a+}^{\alpha}u\right\|_{L^p(a,b)}. \tag{6.17}$$

Proof. From $a<x<b$ we have $\frac{1}{b}<\frac{1}{x}<\frac{1}{a}$. By using Theorem 6.12, we calculate

$$\begin{aligned}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} &= \left(\int_a^b x^{-p}|u(x)|^p dx\right)^{\frac{1}{p}} \\ &\leq a^{-1}\|u\|_{L^p(a,b)} \\ &\overset{(6.9)}{\leq} \frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathcal{D}_{a+}^{\alpha}u\right\|_{L^p(a,b)},\end{aligned} \tag{6.18}$$

showing (6.17). ◻

Theorem 6.19. *Also, Theorem 6.18 holds for the Riemann-Liouville derivative.*

Proof. The proof is similar with Theorem 6.14. ◻

Let us give the weighted one-dimensional Hardy type inequality.

Theorem 6.20. *Let $a>0$, $u\in L^p(a,b)$, $u(a)=0$ and $\mathcal{D}_{a+}^{\alpha}u \in L^p(a,b)$ with $p>1$. Then for the $\mathcal{D}_{a+}^{\alpha}$ Caputo fractional derivative of order $\alpha \in \left(\frac{1}{p},1\right]$ and $\gamma\in\mathbb{R}$, there exists $C>0$ such that*

$$\left\|\frac{u}{x^{\gamma+1}}\right\|_{L^p(a,b)} \leq \frac{a^{-|\gamma|-1}b^{|\gamma|}(b-a)^{\alpha}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)} \left\|\frac{\mathcal{D}_{a+}^{\alpha}u}{x^{\gamma}}\right\|_{L^p(a,b)}. \tag{6.19}$$

Proof. Let us divide the proof in two cases $\gamma \geq 0$ and $\gamma < 0$. Firstly, let us prove the case $\gamma \geq 0$. From $a > 0$, we have $b^{-\gamma-1} < x^{-\gamma-1} < a^{-\gamma-1}$, so that

$$\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &\leq a^{-\gamma-1}\left(\int_a^b |u(x)|^p dx\right)^{\frac{1}{p}} \\
&\overset{(6.9)}{\leq} \frac{a^{-\gamma-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \\
&= \frac{a^{-\gamma-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \qquad (6.20)\\
&\leq \frac{a^{-\gamma-1}b^{\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathcal{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}} dx\right)^{\frac{1}{p}} \\
&= \frac{a^{-\gamma-1}b^{\gamma}(b-a)^{\alpha}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left\|\frac{\mathcal{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p(a,b)}.
\end{aligned}$$

Let us show the case $\gamma < 0$,

$$\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &= \left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma p+p)}}dx\right)^{\frac{1}{p}} \\
&\leq b^{-\gamma}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} \\
&\overset{(6.17)}{\leq} \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \\
&= \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \qquad (6.21)\\
&= \frac{a^{-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathcal{D}^{\alpha}_{a+}u|^p dx\right)^{\frac{1}{p}} \\
&\leq \frac{a^{\gamma-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathcal{D}^{\alpha}_{a+}u|^p}{x^{\gamma p}} dx\right)^{\frac{1}{p}} \\
&= \frac{a^{\gamma-1}b^{-\gamma}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\frac{\mathcal{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p(a,b)},
\end{aligned}$$

implying (6.19). ∎

Remark 6.21. *Also, Theorem 6.20 holds for the Riemann-Liouville derivative.*

6.3. **Gagliardo-Nirenberg type inequality for the Caputo fractional derivative.** Now, we are on a way to establish the Gagliardo-Nirenberg inequality for differential operators of the fractional order. We show that the Sobolev type inequality that is given by Theorem 6.15 implies a family of the Gagliardo–Nirenberg inequalities.

Theorem 6.22. *Assume that* $\alpha \in \left(\frac{1}{q}, 1\right]$, $1 \le p, q < \infty$. *Then we have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^{\gamma}(a,b)} \le C\|\mathcal{D}^{\alpha}_{a+}u\|^{s}_{L^{q}(a,b)}\|u\|^{1-s}_{L^{p}(a,b)}, \tag{6.22}$$

with

$$\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1, \tag{6.23}$$

where $s \in [0, 1]$.

Proof. By using the Hölder inequality with $\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1$, we have

$$\begin{aligned}\int_a^b |u(x)|^{\gamma}dx &= \int_a^b |u(x)|^{\gamma s}|u(x)|^{\gamma(1-s)}dx \\ &\le \left(\int_a^b |u(x)|^q dx\right)^{\frac{\gamma s}{q}} \left(\int_a^b |u(x)|^p dx\right)^{\frac{\gamma(1-s)}{p}} \\ &\overset{(6.9)}{\le} C\|\mathcal{D}^{\alpha}_{a+}u\|^{\gamma s}_{L^q(a,b)}\|u\|^{\gamma(1-s)}_{L^p(a,b)},\end{aligned} \tag{6.24}$$

showing (6.22). $\square$

Remark 6.23. *Also, Theorem 6.22 holds for the Riemann-Liouville derivative.*

Let us consider the space $\dot{H}^{\alpha}_{+}(a, b)$ with $\alpha \in \left(\frac{1}{2}, 1\right]$ in the following form:

$$\dot{H}^{\alpha}_{+}(a, b) := \{u \in L^2(a, b),\ \mathcal{D}^{\alpha}_{a+}u \in L^2(a, b),\ u(a) = 0\}.$$

A special case of Theorem 6.22 important for our further analysis is that of $q = 2$ and $\alpha = 1$, in which case we obtain a more classical Gagliardo-Nirenberg inequality:

Corollary 6.24. *We have the following Gagliardo-Nirenberg type inequality*

$$\|u\|_{L^{\gamma}(a,b)} \le C\|u\|^{s}_{\dot{H}^{1}_{+}(a,b)}\|u\|^{1-s}_{L^{p}(a,b)}, \tag{6.25}$$

for $s \in [0, 1]$.

We also record another more general special case of Theorem 6.22 with $q = 2$:

Corollary 6.25. *Let* $\alpha \in \left(\frac{1}{2}, 1\right]$. *We have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^{\gamma}(a,b)} \lesssim \|u\|^{s}_{\dot{H}^{\alpha}_{+}(a,b)}\|u\|^{1-s}_{L^{p}(a,b)}, \tag{6.26}$$

for $\frac{1}{\gamma} = \frac{s}{2} + \frac{1-s}{p}$, *where* $s \in [0, 1]$.

6.4. **Caffarelli-Kohn-Nirenberg type inequality for the Caputo fractional derivative.** Then let us now show a fractional Caffarelli-Kohn-Nirenberg type inequality.

Theorem 6.26. *Assume that $a > 0$, $\alpha \in \left(1 - \frac{1}{q}, 1\right)$, $1 < p, q < \infty$, $0 < r < \infty$, and $p + q \geq r$. Let $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $c, d, e \in \mathbb{R}$ with the $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, $c = \delta(d-1) + e(1-\delta)$ and $u(a) = 0$. If $1 + (d-1)p > 0$ then we have*

$$\|x^c u\|_{L^r(a,b)} \leq C \|x^d \mathcal{D}^{\alpha}_{a+} u\|^{\delta}_{L^p(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}. \tag{6.27}$$

Proof. Case $\delta = 0$.

If $\delta = 0$, then $c = e$ and $q = r$. Then (6.27) is the inequality

$$\|x^c u\|_{L^r(a,b)} \leq \|x^c u\|_{L^r(a,b)}.$$

Case $\delta = 1$.

If $\delta = 1$, then we have $c = d - 1$ and $p = r$. Also, we have $1 + cp = 1 + (d-1)p > 0$. Then by using weighted fractional Hardy inequality (Theorem 6.20), we obtain

$$\begin{aligned} \|x^c u\|_{L^p(a,b)} &\leq C \left\| x^{c+1} \mathcal{D}^{\alpha}_{a+} u \right\|_{L^p(a,b)} \\ &= C \left\| x^d \mathcal{D}^{\alpha}_{a+} u \right\|_{L^p(a,b)}. \end{aligned} \tag{6.28}$$

Case $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$.

By assumption $c = \delta(d-1) + e(1-\delta)$ and by using Hölder's inequality with $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, we calculate

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &= \left(\int_a^b x^{cr} |u(x)|^r dx \right)^{\frac{1}{r}} \\ &= \left(\int_a^b \frac{|u(x)|^{\delta r}}{x^{\delta r(1-d)}} \frac{|u(x)|^{(1-\delta)r}}{x^{-er(1-\delta)}} dx \right)^{\frac{1}{r}} \\ &\leq \left\| \frac{u}{x^{1-d}} \right\|^{\delta}_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)}. \end{aligned} \tag{6.29}$$

By using weighted fractional Hardy inequality (Theorem 6.20) with $1 + (d-1)p > 0$, we obtain

$$\begin{aligned} \|x^c u\|_{L^r(a,b)} &\leq \left\| \frac{u}{x^{1-d}} \right\|^{\delta}_{L^p(a,b)} \left\| \frac{u}{x^{-e}} \right\|^{1-\delta}_{L^q(a,b)} \\ &\leq C \|x^d \mathcal{D}^{\alpha}_{a+} u\|^{\delta}_{L^p(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}, \end{aligned} \tag{6.30}$$

completing the proof. □

Remark 6.27. *Also, Theorem 6.26 holds for the Riemann-Liouville derivative.*

6.5. **Sequential Derivation Case.** In this subsection we collect results for the sequential derivatives. Indeed, it is important due to the non–commutativity and the absence of the semi–group property of fractional differential operators.

6.6. **Fractional Poincaré–Sobolev type inequality for sequential fractional derivative.**

Theorem 6.28. *Let* $\mathcal{D}^{\beta}_{a+}u(a)=0$, $\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\in L^p(a,b)$ *with* $\alpha\in\left(\frac{1}{q},1\right)$ *and* $\beta\in(0,1)$. *Then the following inequality is true*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)}\leq\frac{(b-a)^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}\tag{6.31}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We consider the function

$$\mathcal{D}^{\beta}_{a+}u(t)=I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(t).\tag{6.32}$$

Using the Hölder inequality

$$\begin{aligned}\left|I^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(t)\right|&\leq\frac{1}{\Gamma(\alpha)}\int_a^t\left|(t-s)^{\alpha-1}\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right|ds\\&\leq\frac{1}{\Gamma(\alpha)}\left(\int_a^t(t-s)^{\alpha q-q}ds\right)^{\frac{1}{q}}\left(\int_a^t\left|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}}\\&=\frac{(t-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left(\int_a^t\left|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u(s)\right|^p ds\right)^{\frac{1}{p}}\\&\leq\frac{(b-a)^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}.\end{aligned}$$

Then we have

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\infty(a,b)}\leq\frac{(b-a)^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)},$$

completing proof. ∎

Remark 6.29. *In Theorem 6.28, if* $1<\theta<\infty$, *then we have*

$$\|\mathcal{D}^{\beta}_{a+}u\|_{L^\theta(a,b)}\leq\frac{(b-a)^{\alpha-\frac{1}{p}+\frac{1}{\theta}}}{(\alpha q-q+1)^{\frac{1}{q}}\,\Gamma(\alpha)}\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}.$$

6.7. **Fractional Hardy type inequality for the sequential fractional derivative.** Now we show the following sequential fractional Hardy inequality.

Theorem 6.30. *Let* $a>0$, $\gamma\in\mathbb{R}$ *and* $\mathcal{D}^{\beta}_{a+}u(a)=0$ *and* $\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\in L^p(a,b)$ *with* $\alpha\in\left(\frac{1}{q},1\right)$. *Then the following inequality is true*

$$\left\|\frac{\mathcal{D}^{\beta}_{a+}u}{x}\right\|_{L^p(a,b)}\leq C\left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)}\tag{6.33}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From $a < x < b$ we have $\frac{1}{b} < \frac{1}{x} < \frac{1}{a}$. By using Theorem 6.28, we calculate

$$\begin{aligned}
\left(\int_a^b \frac{|\mathcal{D}^{\beta}_{a+}u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} &= \left(\int_a^b x^{-p}|\mathcal{D}^{\beta}_{a+}u(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq a^{-1}\|\mathcal{D}^{\beta}_{a+}u\|_{L^p(a,b)} \\
&\overset{(6.31)}{\leq} \frac{a^{-1}(b-a)^{\alpha}}{(\alpha q - q + 1)^{\frac{1}{q}}\,\Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u\right\|_{L^p(a,b)},
\end{aligned} \tag{6.34}$$

showing (6.33). ☐

6.8. **Fractional Gagliardo-Nirenberg type inequality for the sequential fractional derivative.** In the same way as Theorem 6.22 is proved, we can prove the following statement.

Theorem 6.31. *Assume that $1 \leq p, q < \infty$, and let $\alpha \in (0,1)$ be such that $\beta \in \left(\frac{1}{q}, 1\right)$. Suppose that $\mathcal{D}^{\alpha}_{a+}\mathcal{D}^{\beta}_{a+}u \in L^q(a,b)$ and $\mathcal{D}^{\alpha}_{a}u \in L^p(a,b)$. Then we have the following Gagliardo-Nirenberg type inequality,*

$$\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx \lesssim \left(\int_a^b |\mathcal{D}^{\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}}, \tag{6.35}$$

with

$$\frac{s\gamma}{q} + \frac{(1-s)\gamma}{p} = 1, \tag{6.36}$$

where $s \in [0,1]$.

Proof. Let us calculate the following integral:

$$\begin{aligned}
\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx &= \int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{s\gamma}|\mathcal{D}^{\alpha}_{a+}u(x)|^{(1-s)\gamma}dx \\
&\leq \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}},
\end{aligned} \tag{6.37}$$

with

$$\frac{s\gamma}{q} + \frac{(1-s)\gamma}{p} = 1. \tag{6.38}$$

Then by using Theorem 6.28, we obtain

$$\begin{aligned}
\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^{\gamma}dx &\leq \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}} \\
&\overset{(6.33)}{\leq} C\left(\int_a^b |\mathcal{D}^{\beta}_{a+}\mathcal{D}^{\alpha}_{a+}u(x)|^q dx\right)^{\frac{s\gamma}{q}} \left(\int_a^b |\mathcal{D}^{\alpha}_{a+}u(x)|^p dx\right)^{\frac{(1-s)\gamma}{p}}.
\end{aligned} \tag{6.39}$$

The theorem is proved. ☐

6.9. **Poincaré–Sobolev type inequality for the Hadamard fractional derivative.** In this subsection we show fractional order Poincaré–Sobolev type inequality.

Theorem 6.32. *Let $a>0$, $u\in L^p(a,b)$, $u(a)=0$, $\mathfrak{D}^\alpha_{a+}u\in L^p_{\frac{1}{x}}(a,b)$ and $p>1$. Then for the $\mathfrak{D}^\alpha_{a+}$ Hadamard fractional derivative of order $\alpha\in\left(\frac{1}{p},1\right]$ we have the inequality*

$$\|u\|_{L^\infty(a,b)}\le\frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.40}$$

Proof. Let $u\in L^p_{\frac{1}{x}}(a,b)$, $u(a)=0$, $\mathfrak{D}^\alpha_{a+}u\in L^p(a,b)$ and consider the function

$$u(t)=\mathfrak{I}^\alpha_{a+}\mathfrak{D}^\alpha_{a+}u(t). \tag{6.41}$$

Using the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}
\left|\mathfrak{I}^\alpha_{a+}\mathfrak{D}^\alpha_{a+}u(t)\right| &\le\frac{1}{\Gamma(\alpha)}\int_a^t\left|\left(\log\frac{t}{s}\right)^{\alpha-1}\mathfrak{D}^\alpha_{a+}u(s)\right|\frac{ds}{s^{\frac{1}{p}+\frac{1}{q}}}\\
&\le\frac{1}{\Gamma(\alpha)}\left(\int_a^t\left|\log\frac{t}{s}\right|^{\alpha q-q}\frac{ds}{s}\right)^{\frac{1}{q}}\left(\int_a^t\left|\mathfrak{D}^\alpha_{a+}u(s)\right|^p\frac{ds}{s}\right)^{\frac{1}{p}}\\
&\overset{\alpha>\frac{1}{p}}{=}\frac{\left|\log\frac{t}{a}\right|^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left(\int_a^t\left|\mathfrak{D}^\alpha_{a+}u(s)\right|^p\frac{ds}{s}\right)^{\frac{1}{p}}\\
&\le\frac{\left|\log\frac{b}{a}\right|^{\alpha-1+\frac{1}{q}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}\\
&=\frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{(\alpha q-q+1)^{\frac{1}{q}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}\\
&=\frac{\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)},
\end{aligned}$$

where $q=\frac{p}{p-1}>1$, showing (6.40).

□

Remark 6.33. *In Theorem 6.32, by taking $1<\theta<\infty$, we have*

$$\|u\|_{L^\theta(a,b)}\le\frac{(b-a)^{\frac{1}{\theta}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.42}$$

6.10. **Hardy type inequality for the Hadamard fractional derivative.** Let us show Hardy inequality.

Theorem 6.34. *Let $a>0$, $u(a)=0$ and $\mathfrak{D}^{\alpha}_{a+}u\in L^p_{\frac{1}{x}}(a,b)$ with $p>1$. Then for the $\mathfrak{D}^{\alpha}_{a+}$ Hadamard fractional derivative of order $\alpha\in\left(\frac{1}{p},1\right]$ we have the inequality*

$$\left\|\frac{u}{x}\right\|_{L^p(a,b)}\leq\frac{a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^{\alpha}_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.43}$$

Proof. From $a<x<b$ we have $\frac{1}{b}<\frac{1}{x}<\frac{1}{a}$. By using Theorem 6.32, we calculate

$$\begin{aligned}\left(\int_a^b\frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}}&=\left(\int_a^b x^{-p}|u(x)|^p dx\right)^{\frac{1}{p}}\\&\leq a^{-1}\|u\|_{L^p(a,b)}\\&\overset{(6.40)}{\leq}\frac{a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^{\alpha}_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)},\end{aligned} \tag{6.44}$$

showing (6.43). ∎

Let us show weighted Hardy inequality with Hadamard derivaive.

Theorem 6.35. *Let $a>0$, $u(a)=0$ and $\mathfrak{D}^{\alpha}_{a+}u\in L^p_{\frac{1}{x}}(a,b)$ with $p>1$. Then for the $\mathfrak{D}^{\alpha}_{a+}$ Hadamard fractional derivative of order $\alpha\in\left(\frac{1}{p},1\right]$ and $\gamma\in\mathbb{R}$, we have inequality*

$$\left\|\frac{u}{x^{\gamma+1}}\right\|_{L^p(a,b)}\leq C\left\|\frac{\mathfrak{D}^{\alpha}_{a+}u}{x^{\gamma}}\right\|_{L^p_{\frac{1}{x}}(a,b)}. \tag{6.45}$$

Proof. Let us divide the proof in two cases $\gamma \geq 0$ and $\gamma < 0$. Firstly, let us prove the case $\gamma \geq 0$. From $a > 0$, we have $b^{-\gamma-1} < x^{-\gamma-1} < a^{-\gamma-1}$

$$\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &\leq a^{-\gamma-1}\left(\int_a^b |u(x)|^p dx\right)^{\frac{1}{p}} \\
&\overset{(6.40)}{\leq} \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathfrak{D}^\alpha_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\
&= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathfrak{D}^\alpha_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \qquad (6.46)\\
&\leq \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}b^\gamma\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathfrak{D}^\alpha_{a+}u|^p}{x^{\gamma p}}\frac{dx}{x}\right)^{\frac{1}{p}} \\
&= \frac{a^{-\gamma-1}(b-a)^{\frac{1}{p}}b^\gamma\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\frac{\mathfrak{D}^\alpha_{a+}u}{x^\gamma}\right\|_{L^p_{\frac{1}{x}}(a,b)}.
\end{aligned}$$

Let us show the case $\gamma < 0$,

$$\begin{aligned}
\left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma+1)p}}dx\right)^{\frac{1}{p}} &= \left(\int_a^b \frac{|u(x)|^p}{x^{(\gamma p+p)}}dx\right)^{\frac{1}{p}} \\
&\leq b^{-\gamma}\left(\int_a^b \frac{|u(x)|^p}{x^p}dx\right)^{\frac{1}{p}} \\
&\overset{(6.43)}{\leq} \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\mathfrak{D}^\alpha_{a+}u\right\|_{L^p_{\frac{1}{x}}(a,b)} \\
&= \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b |\mathfrak{D}^\alpha_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \qquad (6.47)\\
&= \frac{b^{-\gamma}a^{-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{x^{\gamma p}}{x^{\gamma p}}|\mathfrak{D}^\alpha_{a+}u|^p\frac{dx}{x}\right)^{\frac{1}{p}} \\
&\leq \frac{b^{-\gamma}a^{\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left(\int_a^b \frac{|\mathfrak{D}^\alpha_{a+}u|^p}{x^{\gamma p}}\frac{dx}{x}\right)^{\frac{1}{p}} \\
&= \frac{b^{-\gamma}a^{\gamma-1}(b-a)^{\frac{1}{p}}\left|\log\frac{b}{a}\right|^{\alpha-\frac{1}{p}}}{\left(\frac{\alpha p}{p-1}-\frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)}\left\|\frac{\mathfrak{D}^\alpha_{a+}u}{x^\gamma}\right\|_{L^p_{\frac{1}{x}}(a,b)},
\end{aligned}$$

showing (6.45). □

6.11. **Fractional Gagliardo-Nirenberg type inequality with the Hadamard derivative.**

Theorem 6.36. *Assume that* $\alpha \in \left(\frac{1}{q}, 1\right]$, $1 \leq p, q < \infty$. *Then we have the following Gagliardo-Nirenberg type inequality,*

$$\|u\|_{L^\gamma(a,b)} \leq C \|\mathfrak{D}^\alpha_{a+} u\|^s_{L^q_{\frac{1}{x}}(a,b)} \|u\|^{1-s}_{L^p(a,b)}, \tag{6.48}$$

with

$$\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1, \tag{6.49}$$

where $s \in [0,1]$.

Proof. By using the Hölder inequality $\frac{\gamma s}{q} + \frac{\gamma(1-s)}{p} = 1$, we have

$$\begin{aligned} \int_a^b |u(x)|^\gamma dx &= \int_a^b |u(x)|^{\gamma s} |u|^{\gamma(1-s)} dx \\ &\leq \left(\int_a^b |u(x)|^q dx \right)^{\frac{\gamma s}{q}} \left(\int_a^b |u(x)|^p dx \right)^{\frac{\gamma(1-s)}{p}} \\ &\overset{(6.40)}{\leq} C \|\mathfrak{D}^\alpha_{a+} u\|^{\gamma s}_{L^q_{\frac{1}{x}}(a,b)} \|u\|^{\gamma(1-s)}_{L^p(a,b)}, \end{aligned} \tag{6.50}$$

completing proof. □

6.12. **Fractional Cafarrelli-Kohn-Nirenberg type inequality with Hadamard derivative.** Then let us show fractional Cafarrelli-Kohn-Nirenberg type inequality.

Theorem 6.37. *Assume that* $a > 0$, $\alpha \in \left(1 - \frac{1}{q}, 1\right)$, $1 < p, q < \infty$, $0 < r < \infty$ *such that* $p + q \geq r$. *Let* $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ *and* $c, d, e \in \mathbb{R}$ *with the* $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, $c = \delta(d-1) + e(1-\delta)$ *and* $u(a) = 0$. *If* $1 + (d-1)p > 0$ *then we have*

$$\|x^c u\|_{L^r(a,b)} \leq C \left\| x^d \mathfrak{D}^\alpha_{a+} u \right\|^\delta_{L^p_{\frac{1}{x}}(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)}. \tag{6.51}$$

Proof. Case $\delta = 0$.

If $\delta = 0$, then $c = e$ and $q = r$. Then (6.27) is the inequality

$$\|x^c u\|_{L^r(a,b)} \leq \|x^c u\|_{L^r(a,b)}.$$

Case $\delta = 1$.

If $\delta = 1$, then we have $c = d - 1$ and $p = r$. Also, we have $1 + cp = 1 + (d-1)p > 0$. Then by using weighted fractional Hardy inequality (Theorem 6.35) we obtain

$$\begin{aligned} \|x^c u\|_{L^p(a,b)} &\leq C \left\| x^{c+1} \mathfrak{D}^\alpha_{a+} u \right\|_{L^p_{\frac{1}{x}}(a,b)} \\ &= C \left\| x^d \mathfrak{D}^\alpha_{a+} u \right\|_{L^p_{\frac{1}{x}}(a,b)}. \end{aligned} \tag{6.52}$$

Case $\delta \in [0,1] \cap [\frac{r-q}{r}, \frac{p}{r}]$.

By assumption $c = \delta(d-1) + e(1-\delta)$ and by using Hölder's inequality with $\frac{\delta}{p} + \frac{1-\delta}{q} = \frac{1}{r}$, we calculate

$$\begin{aligned}\|x^c u\|_{L^r(a,b)} &= \left(\int_a^b x^{cr}|u(x)|^r dx\right)^{\frac{1}{r}} \\ &= \left(\int_a^b \frac{|u(x)|^{\delta r}}{x^{\delta r(1-d)}} \frac{|u(x)|^{(1-\delta)r}}{x^{-er(1-\delta)}} dx\right)^{\frac{1}{r}} \\ &\leq \left\|\frac{u}{x^{1-d}}\right\|^{\delta}_{L^p(a,b)} \left\|\frac{u}{x^{-e}}\right\|^{1-\delta}_{L^q(a,b)}.\end{aligned} \tag{6.53}$$

By using weighted fractional Hardy inequality (Theorem 6.35) with $1 + (d-1)p > 0$ we obtain

$$\begin{aligned}\|x^c u\|_{L^r(a,b)} &\leq \left\|\frac{u}{x^{1-d}}\right\|^{\delta}_{L^p(a,b)} \left\|\frac{u}{x^{-e}}\right\|^{1-\delta}_{L^q(a,b)} \\ &\leq C\|x^d \mathfrak{D}^{\alpha}_{a+} u\|^{\delta}_{L^p_{\frac{1}{x}}(a,b)} \|x^e u\|^{1-\delta}_{L^q(a,b)},\end{aligned} \tag{6.54}$$

showing (6.51). □

6.13. **Lyapunov-type inequality.** Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open bounded domain, $-\infty < a < b < +\infty$ and $q(x)$ be real-valued, continuous function. Let us consider the following fractional differential equation:

$$\begin{cases} \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}u(x,y) - (-\Delta_y)^s_p u(x,y) + q(x)u(x,y) = 0, \text{ in } (a,b)\times\Omega, \\ u(a,y) = u(b,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.55}$$

where $\mathcal{D}^{\mu}_{a+,x}$ is the Caputo fractional derivative in the variable x and $(-\Delta_y)^s_p$ is the fractional p-Laplacian in the variable y with $s \in (0,1)$ and $1 < p < \infty$.

By [125], we can choose the first eigenfunction of

$$\begin{cases} (-\Delta_y)^s_p \varphi_1(y) = \lambda_1(\Omega)\varphi_1(y), \quad y \in \Omega, \\ \varphi_1(y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.56}$$

corresponding to be positive and whose eigenvalue simple and positive, $\lambda_1(\Omega) > 0$.

In this section we obtain a Lyapunov-type inequality for (6.55).

Theorem 6.38. *Assume that $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha + \beta \leq 2$, $s \in (0,1)$, $1 < p < \infty$ and $q(x) \in C([a,b])$. Then for (6.55), we have*

$$\int_a^b |q(x) - \lambda_1(\Omega)| dx \geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \tag{6.57}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of (6.56).

Proof. By multiplying (6.55) with $\varphi_1(y)$ and integrating over Ω, we get

$$\begin{aligned}\int_\Omega \mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}u(x,y)\varphi_1(y)dy - &\int_\Omega((-\Delta_y)^s_p u(x,y))\varphi_1(y)dy\\ &+q(x)\int_\Omega u(x,y)\varphi_1(y)dy\\ &= \mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy\\ &-\int_\Omega((-\Delta_y)^s_p u(x,y))\varphi_1(y)dy\\ &+q(x)\int_\Omega u(x,y)\varphi_1(y)dy\\ &= \mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy\\ &-\int_\Omega((-\Delta_y)^s_p \varphi_1(y))u(x,y)dy\\ &+q(x)\int_\Omega u(x,y)\varphi_1(y)dy\\ &= \mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}\int_\Omega u(x,y)\varphi_1(y)dy\\ &-\lambda_1(\Omega)\int_\Omega u(x,y)\varphi_1(y)dy\\ &+\int_\Omega u(x,y)\varphi_1(y)dy\\ &= \mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}v(x)+q_1(x)v(x)=0,\end{aligned}$$

where $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$, $q_1(x) = q(x) - \lambda_1(\Omega)$, from boundary condition (6.55), we have

$$v(a)=0,\ v(b)=0.$$

Finally, we get

$$\begin{cases}\mathcal{D}^\alpha_{a+,x}\mathcal{D}^\beta_{a+,x}v(x)+q_1(x)v(x)=0, & x\in(a,b),\\ v(a)=0,\ v(b)=0.\end{cases} \tag{6.58}$$

By [126], for the (6.58), we get

$$\begin{aligned}\int_a^b |q_1(x)|dx &= \int_a^b |q(x)-\lambda_1(\Omega)|dx\\ &\geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta},\end{aligned} \tag{6.59}$$

completing the proof. □

Corollary 6.39. *By choosing $\alpha=\beta=1$, $s=1$ and $p=2$, we have Theorem 2.2 in [121] with $\gamma=0$.*

Let us consider the following eigenvalue problem in cylindrical domain:

$$\begin{cases} \mathcal{D}^{\alpha}_{a+,x}\mathcal{D}^{\beta}_{a+,x}u(x,y) - (-\Delta_y)^s u(x,y) + \nu u(x,y) = 0, \text{ in } (a,b)\times\Omega, \\ u(a,y) = u(b,y) = 0, \quad \forall y \in \Omega, \\ u(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.60}$$

where $(-\Delta_y)^s$ is the fractional Laplacian. Denote that $|\cdot|$ is the Lebesgue measure. Then, we have the following two sides estimate of the first eigenvalue of (6.60) in the circular cylinder.

Theorem 6.40. *Let $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha+\beta \leq 2$, $s \in (0,1)$ and $1 < p < \infty$. Then we have,*

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &> \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \end{aligned} \tag{6.61}$$

where $\lambda_1(B)$ is the first eigenvalue of the eigenvalue problem (6.60) in a ball B with $|\Omega| = |B|$.

Proof. By using previous theorem, assume that B be a ball and $q(x) = \nu$ by using Theorem A.1 in [1], we have

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &\geq \int_a^b |\nu - \lambda_1(B)| dx \\ &\geq \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}. \end{aligned} \tag{6.62}$$

□

Theorem 6.41. *Assume that $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha+\beta \leq 2$, $s \in (0,1)$ and $1 < p < \infty$. Then we have,*

$$\begin{aligned} (b-a)|\nu| + (b-a)\lambda_1(\Omega) &\geq (b-a)|\nu| + (b-a)\lambda_1(B) \\ &> \frac{\Gamma(\alpha+\beta)(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(b-a)^{\alpha+\beta-1}(\alpha+\beta-1)^{\alpha+\beta-1}\beta^{\beta}}, \end{aligned} \tag{6.63}$$

where $\lambda_1(B)$ is the first eigenvalue of the eigenvalue problem (6.60) in ball B with $|\Omega| = |B|$.

Let us consider the following fractional differential equation by the variable x:

$$\begin{cases} L_x u(x,y) - (-\Delta_y)^s u(x,y) + q(x)u(x,y) = 0, \quad (x,y) \in (a,b)\times\Omega, \\ u(a,y) = u(b,y) = 0, \quad \forall y \in \Omega, \\ u(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.64}$$

where

$$L_x := \frac{D^{\alpha}_{b-,x} D^{\alpha}_{a+,x} + D^{\alpha}_{a+,x} D^{\alpha}_{b-,x}}{2}, \quad \frac{1}{2} < \alpha < 1.$$

Denote that the following functional spaces $AC^{\alpha,2}_a([a,b])$ and $AC^{\alpha,2}_b([a,b])$.

Definition 6.42. For all $\alpha \in (0,1)$ and every $1 \leq p < \infty$, we denote by $AC_a^{\alpha,2}([a,b])$ the functional space defined by

$$AC_a^{\alpha,2}([a,b]) := \{f : f \in L^1([a,b]),\ D_{a+,x}^{\alpha} f \in L^2([a,b])\}. \tag{6.65}$$

$$AC_b^{\alpha,2}([a,b]) := \{f : f \in L^1([a,b]),\ D_{b-,x}^{\alpha} f \in L^2([a,b])\}. \tag{6.66}$$

Then, we have the following Lyapunov-type inequality:

Theorem 6.43. *Suppose that $\frac{1}{2} < \alpha < 1$, $s \in (0,1)$, $1 < p < \infty$ and $q \in C([a,b])$. Suppose that $\int_\Omega u(x,y)\varphi_1(y)dy \in AC_a^{\alpha,2}([a,b]) \cap AC_b^{\alpha,2}([a,b]) \cap C([a,b])$. Then for (6.64), we get*

$$\int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \Gamma^2(\alpha)\left(\frac{2}{b-a}\right)^{2\alpha-1}(2\alpha-1), \tag{6.67}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of (6.56).

Proof. The proof is similar to that of Theorem 6.38. Shortly, we have

$$\begin{cases} L_x v(x) + q_1(x)v(x) = 0, \quad x \in (a,b), \\ v(a) = 0,\ v(b) = 0, \end{cases} \tag{6.68}$$

where $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$ and $q_1(x) = q(x) - \lambda_1(\Omega)$. By assumptions $v(x) \in AC_a^{\alpha,2}([a,b]) \cap AC_b^{\alpha,2}([a,b]) \cap C([a,b])$ and from [127], we get

$$\int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \Gamma^2(\alpha)\left(\frac{2}{b-a}\right)^{2\alpha-1}(2\alpha-1). \tag{6.69}$$

Theorem 6.43 is complete. □

6.14. **Hartman-Wintner-type inequality.** In this section, we show Hartman-Wintner type inequality for problem (6.55).

Theorem 6.44. *Assume that $0 < \alpha, \beta \leq 1$ be such that $1 < \alpha + \beta \leq 2$, $s \in (0,1)$, $1 < p < \infty$ and $q(x) \in C([a,b])$. Assume that the fractional boundary value problem (6.55) has a nontrivial continuous solution. Then, we have*

$$\int_a^b (b-s)^{\alpha+\beta-1}(s-a)^{\beta}[q(x) - \lambda_1(\Omega)]^+ ds > \Gamma(\alpha+\beta)(b-a)^{\beta}, \tag{6.70}$$

where $[q(x) - \lambda_1(\Omega)]^+ = \max\{q(x) - \lambda_1(\Omega), 0\}$.

Proof. By multiplying (6.55) with $\varphi_1(y)$ and integrating over Ω, for the function $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$ we have problem (6.58). Problem (6.58) is equivalent to the integral equation (see. [126])

$$v(x) = \int_a^b G(x,s)q_1(s)v(s)ds,$$

where

$$G(x,s)=\frac{1}{\Gamma(\alpha+\beta)}\begin{cases}\frac{(b-s)^{\alpha+\beta-1}(x-a)^{\beta}}{(b-a)^{\beta}}-(x-s)^{\alpha+\beta-1}, & a\le s\le x\le b,\\ \\ \frac{(b-s)^{\alpha+\beta-1}(x-a)^{\beta}}{(b-a)^{\beta}}, & a\le x\le s\le b,\end{cases}\tag{6.71}$$

$$G(x,s)\le G(s,s),\ \text{ for } x,s\in[a,b].\tag{6.72}$$

By using last fact with (6.72) for any $a\le x\le b$, we get

$$\begin{aligned}|v(x)| &\le \int_a^b |G(x,s)||q_1(s)||v(s)|ds\\ &\le \int_a^b G(s,s)|q_1(s)||v(s)|ds\\ &\le \frac{(b-a)^{-\beta}}{\Gamma(\alpha+\beta)}\int_a^b (b-s)^{\alpha+\beta-1}(s-a)^{\beta}q_1^{+}(s)|v(s)|ds.\end{aligned}$$

Theorem 6.70 is proved. □

Corollary 6.45. *By choosing $\alpha=\beta=1$ and $s=1$, $p=2$ in (6.70), we have the classical Hartman-Wintner inequality*

$$\int_a^b (b-s)(s-a)q_1^{+}(s)>b-a.\tag{6.73}$$

6.15. **De La Vallée Poussin-type inequality.** Let us consider in $(a,b)\times\Omega$ the following fractional differential Dirichlet problem:

$$\begin{cases}\frac{\partial^2}{\partial x^2}u(x,y)-(-\Delta_y)^s u(x,y)+f(x)\mathcal{D}^{\alpha}_{a+,x}u(x,y)+q(x)u(x,y)=0,\\ u(a,y)=u(b,y)=0, \quad \forall y\in\Omega,\\ u(x,y)=0, \quad y\in\mathbb{R}^N\setminus\Omega,\end{cases}\tag{6.74}$$

where $\alpha\in(0,1]$. Then, we show a de La Vallée Poussin-type inequality for (6.74).

Theorem 6.46. *Assume that $\alpha\in(0,1]$. Then, for (6.74), we have De La Vallée Poussin-type inequality in the following form:*

$$1<M_1(b-a)^{2-\alpha}+M_2\frac{(b-a)^2}{\Gamma(2-\alpha)},$$

where $M_1=\max_{a\le x\le b}|f(x)|$, $M_2=\max_{a\le x\le b}|q(x)-\lambda_1(\Omega)|$ and $\lambda_1(\Omega)$ is the first eigenvalue of the (6.56).

Proof. Similarly to Theorem 6.46, we get

$$v''(x)+f(x)\mathcal{D}^{\alpha}_{a+,x}v(x)+q_1(x)v(x)=0,\tag{6.75}$$

with

$$v(a)=v(b)=0,$$

where

$$v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$$

and

$$q_1(x) = q(x) - \lambda_1(\Omega).$$

From [120, Theorem 3.1], we have

$$\begin{aligned} b-a &< \max\left\{\int_a^b \frac{(s-a)^{2-\alpha}}{\Gamma(2-\alpha)}|f(s)|ds, \int_a^b \frac{(s-a)^{1-\alpha}}{\Gamma(2-\alpha)}(b-s)|f(s)|ds\right\} \\ &\quad + \int_a^b (s-a)(b-s)|q(s)-\lambda_1(\Omega)|ds \\ &\quad \le M_1(b-a)^{3-\alpha} + M_2(b-a)^3, \end{aligned} \tag{6.76}$$

where $M_1 = \max_{a\le x\le b}|f(x)|$, $M_2 = \max_{a\le x\le b}|q(x)-\lambda_1(\Omega)|$.
Theorem 6.46 is complete. □

Corollary 6.47. *By choosing $\alpha = 1$, we get Theorem 2.2 in* [122].

Let us consider in $(a,b)\times\Omega$ the following fractional differential equation with Riemann-Liouville derivative and $1<\alpha\le 2$ and $0<\beta\le 1$:

$$\begin{cases} D^\alpha_{a+,x}u(x,y) - (-\Delta_y)^s u(x,y) + f(x)D^\beta_{a+,x}u(x,y) + q(x)u(x,y) = 0, \\ u(a,y) = u(b,y) = 0, \quad \forall y \in \Omega, \\ u(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.77}$$

Then let us present de La Vallée Poussin-type inequality for (6.77),

Theorem 6.48. *Assume that $\alpha-\beta \ge 1$ with $1<\alpha\le 2$ and $0<\beta\le 1$. Then, we have*

$$\Gamma(\alpha-\beta) \le C_1M_1 + C_2M_2, \tag{6.78}$$

where $M_1 = \max\limits_{a\le x\le b}|f(x)|$, $M_2 = \max\limits_{a\le x\le b}|q(x)-\lambda_1(\Omega)|$,

$$C_1 = (b-a)^{\alpha-\beta}, \tag{6.79}$$

and

$$C_2 = \frac{(b-a)^\alpha}{\Gamma(1+\beta)}. \tag{6.80}$$

Proof. Similarly to Theorem 6.46, we get

$$D^\alpha_{a+}v(x) + f(x)D^\beta_{a+}v(x) + q_1(x)v(x) = 0, \tag{6.81}$$

with

$$v(a) = v(b) = 0,$$

where

$$v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$$

and

$$q_1(x) = q(x) - \lambda_1(\Omega).$$

By using [120, Theorem 3.11], we have

$$\Gamma(\alpha-\beta) \le C_1'M_1 + C_2'M_2,$$

where

$$C_1' = \max\left\{\int_a^b F_1(s)ds,\ \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}ds\right\}, \tag{6.82}$$

and

$$C_2' = \max\left\{\int_a^b F_2(s)ds,\ \int_a^b \frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(s-a)^{\beta}}{\Gamma(\beta+1)}ds\right\}, \tag{6.83}$$

where

$$F_1(s) = \max\left\{\frac{(s-a)^{\alpha-\beta-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} : \alpha-\beta-1>0,\ (b-s)^{\alpha-\beta-1} - \frac{(b-s)^{\alpha-1}}{(b-a)^{\beta}}\right\}$$

and

$$F_2(s) = F_1(s)\frac{(s-a)^{\beta}}{\Gamma(\beta+1)}.$$

Hence, we get

$$\begin{aligned}\Gamma(\alpha-\beta) &\le C_1'M_1 + C_2'M_2\\ &\le (b-a)^{\alpha-\beta}M_1 + \frac{(b-a)^{\alpha}}{\Gamma(1+\beta)}M_2.\end{aligned}$$

□

Corollary 6.49. *By choosing $\alpha = 2$ and $\beta = 1$ in Theorem 6.48, we get Theorem 2.2 in [122].*

Corollary 6.50. *By choosing $\alpha = 2$ in Theorem 6.48, we get Theorem 6.46.*

6.16. **Lyapunov-type inequality for a fractional differential system.** In this section we present Lyapunov-type inequality for fractional differential system. Let us consider in $(a,b)\times\Omega$ the following fractional differential systems:

$$\begin{cases} u_{xx}(x,y) - (-\Delta_y)^s v(x,y) + f(x)v(x,y) = 0,\\ v_{xx}(x,y) - (-\Delta_y)^s u(x,y) + g(x)u(x,y) = 0,\end{cases} \tag{6.84}$$

with homogeneous Dirichlet problem

$$u(a,y) = u(b,y) = v(a,y) = v(b,y) = 0, \quad y\in\Omega,$$

and

$$u(x,y) = v(x,y) = 0, \quad y \in \mathbb{R}^N\setminus\Omega.$$

Let us show one of the main result of this section:

Theorem 6.51. *Assume that that $f,g\ge 0$ and $f,g\in L^1([a,b])$. If (6.84) has not nontrivial solution, then we have*

$$4 \le (b-a)\left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac12}\left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac12}. \tag{6.85}$$

Proof. Suppose that

$$U(x) = \int_\Omega u(x,y)\varphi_1(y)dy,$$

and

$$V(x) = \int_\Omega v(x,y)\varphi_1(y)dy.$$

Similarly with the single equation case, we have

$$\begin{cases} U''(x) - f_1(x)V(x,y) = 0, \\ V''(x) - g_1(x)U(x,y) = 0, \end{cases} \tag{6.86}$$

with

$$U(a) = U(b) = 0,$$
$$V(a) = V(b) = 0,$$
$$f_1(x) = f(x) - \lambda_1(\Omega)$$

and

$$g_1(x) = g(x) - \lambda_1(\Omega).$$

From [70], we have

$$4 \leq (b-a)\left(\int_a^b |f_1(x)|dx\right)^{\frac{1}{2}} \left(\int_a^b |g_1(x)|dx\right)^{\frac{1}{2}}.$$

Theorem 6.51 is proved. □

Let us consider in $(a,b)\times\Omega$ the following system:

$$\begin{cases} L_x^\alpha u(x,y) - (-\Delta_y)^s v(x,y) + f(x)v(x,y) = 0, \\ L_x^\beta v(x,y) - (-\Delta_y)^s u(x,y) + g(x)u(x,y) = 0, \end{cases} \tag{6.87}$$

with a homogeneous Dirichlet boundary condition

$$u(a,y) = u(b,y) = v(a,y) = v(b,y) = 0, \quad y \in \Omega,$$

and

$$u(x,y) = v(x,y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega,$$

where

$$L_x^\alpha := \frac{D_{b-,x}^\alpha D_{a+,x}^\alpha + D_{a+,x}^\alpha D_{b-,x}^\alpha}{2}, \quad \frac{1}{2} < \alpha < 1,$$

and

$$L_x^\beta := \frac{D_{b-,x}^\beta D_{a+,x}^\beta + D_{a+,x}^\beta D_{b-,x}^\beta}{2}, \quad \frac{1}{2} < \beta < 1.$$

Theorem 6.52. *Suppose that $\frac{1}{2} < \alpha < 1$, $\frac{1}{2} < \beta < 1$ and $f, g \in L^1([a,b])$. Let u, v be a nontrivial solution of (6.87), then we have*

$$\left(\frac{2}{b-a}\right)^{\alpha+\beta-1} (2\alpha-1)^{\frac{1}{2}}(2\beta-1)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(\beta)$$
$$\leq \left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}} \left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}. \tag{6.88}$$

Proof. Proof of this Theorem is similar Theorem 6.51 we obtain

$$\begin{cases} L^{\alpha}U(x) - f_1(x)V(x,y) = 0, \\ L^{\beta}(x) - g_1(x)U(x,y) = 0, \end{cases} \tag{6.89}$$

where

$$U(x) = \int_{\Omega} u(x,y)\varphi_1(y)dy,$$

and

$$V(x) = \int_{\Omega} v(x,y)\varphi_1(y)dy.$$

By using Corollary 5.5 in [127], we get

$$\left(\frac{2}{b-a}\right)^{\alpha+\beta-1}(2\alpha-1)^{\frac{1}{2}}(2\beta-1)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(\beta) \leq \left(\int_a^b |f(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}\left(\int_a^b |g(x)-\lambda_1(\Omega)|dx\right)^{\frac{1}{2}}. \tag{6.90}$$

Theorem 6.52 is proved. □

6.17. **Applications.** In this Section we show some applications of the obtained inequalities and we note that u is a real-valued function.

6.17.1. *Uncertainly principle.* The inequality (6.17) implies the following uncertainly principle:

Corollary 6.53. *Let $a > 0$, $u(a) = 0$ and $\mathcal{D}^{\alpha}_{a+}u \in L^p(a,b)$ with $p > 1$. Then for the Caputo fractional derivative $\mathcal{D}^{\alpha}_{a+}$ of order $\alpha \in \left(\frac{1}{p}, 1\right]$ we have following inequality*

$$\|u\|^2_{L^2(a,b)} \leq \frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \||x|^{\alpha}u\|_{L^q(a,b)}, \tag{6.91}$$

where $q = \frac{p}{p-1}$.

Proof. By using (6.17), we obtain

$$\frac{a^{-1}(b-a)^{\alpha}}{\left(\frac{\alpha p}{p-1} - \frac{1}{p-1}\right)^{\frac{p-1}{p}}\Gamma(\alpha)} \left\|\mathcal{D}^{\alpha}_{a+}u\right\|_{L^p(a,b)} \|xu\|_{L^q(a,b)} \overset{(6.17)}{\geq} \left\|\frac{u}{x}\right\|_{L^p(a,b)} \|xu\|_{L^q(a,b)} \geq \|u\|^2_{L^2(a,b)}, \tag{6.92}$$

completing the proof. □

Remark 6.54. *Also, the uncertainly principle holds for the Riemann-Liouville derivative.*

6.17.2. *Embedding of spaces.* Let us consider the space $\dot{H}^{\alpha}_{+}(a,b)$ with $\alpha \in \left(\frac{1}{2}, 1\right]$ introduced in [128, 129] in the following form:

$$\dot{H}^{\alpha}_{+}(a,b) := \{u \in L^2(a,b),\ \mathcal{D}^{\alpha}_{a+}u \in L^2(a,b),\ u(a) = 0\}.$$

If $\alpha < \beta$, then by Poincaré–Sobolev-type inequality (6.9) we have $\dot{H}^{\beta}_{+}(a,b) \hookrightarrow \dot{H}^{\alpha}_{+}(a,b)$.

6.17.3. *A-priori estimate.* Here, we seek a real-valued solution to the following space-fractional diffusion problem

$$\begin{cases} u_t(x,t) + D^{\alpha}_{b-}\mathcal{D}^{\alpha}_{a+}u(x,t) = 0, & (x,t) \in (a,b) \times (0,T), \\ u(x,0) = u_0(x), \ \ \forall x \in (a,b), \end{cases} \tag{6.93}$$

where $\alpha \in \left(\frac{1}{2}, 1\right]$, $u \in L^{\infty}(0,T; \dot{H}^{\alpha}_{+}(a,b))$, $u_t \in L^2(0,T; \dot{H}^{\alpha}_{+}(a,b))$ and $u_0 \in L^2(a,b)$. We show an a-priori estimate for this problem. Let us define

$$I(t) = \|u(x,\cdot)\|^2_{L^2(a,b)} = \int_a^b |u(x,t)|^2 dx.$$

Then by multiplying (6.93) by u, integrating over (a,b), and by using integration by parts, we compute

$$\begin{aligned} &\int_a^b u_t(x,t)u(x,t)dx + \int_a^b u(x,t)D^{\alpha}_{b-}\mathcal{D}^{\alpha}_{a+}u(x,t)dx \\ &\qquad = \frac{1}{2}\frac{d}{dt}\int_a^b |u(x,t)|^2 dx + \int_a^b |\mathcal{D}^{\alpha}_{a+}u(x,t)|^2 dx \\ &\qquad = \frac{1}{2}\frac{dI(t)}{dt} + \int_a^b |\mathcal{D}^{\alpha}_{a+}u(x,t)|^2 dx. \end{aligned} \tag{6.94}$$

By using (6.9) with $p = 2$ in (6.94), we get

$$0 = \frac{1}{2}\frac{dI(t)}{dt} + \int_a^b |\mathcal{D}^{\alpha}_{a+}u(x,t)|^2 dx \overset{(6.9)}{\geq} \frac{1}{2}\frac{dI(t)}{dt} + \frac{(2\alpha-1)\,\Gamma^2(\alpha)}{(b-a)^{2\alpha}}\int_a^b |u(x,t)|^2 dx, \tag{6.95}$$

it means $\frac{dI(t)}{dt} \leq 0$. That is, $I(t)$ is a non-decreasing function, then for $t > 0$, we have $I(t) \leq I(0)$. Finally,

$$\|u(x,\cdot)\|_{L^2(a,b)} \leq \|u_0\|_{L^2(a,b)}.$$

7. Conclusion

In this PhD dissertation, we develop fractional functional and geometric inequalities on homogeneous Lie groups. More precisely, we develop the fractional calculus and non-commutative analysis, i.e., we combined two big direction in mathematics. This perspective turned out to be extremely useful on both a conceptual and a technical level. Let us review the obtained results in this dissertation:

In Chapter 3, where we study fractional functional and geometric inequalities on homogeneous Lie groups. We obtain fractional Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups and its logarithmic fractional inequalities which is even new on Euclidean case. For the Riesz potential (or a fractional integral), we get the Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups, which means boundedness of the Riesz operator in $L^p - L^q$ spaces. Also, we obtain the Stein-Weiss inequality for the Riesz potential. In addition, we show integer order logarithmic Sobolev-Folland-Stein inequality on stratified Lie groups.

In Chapter 4, where we focus questions of the reverse functional inequalities. We established reverse integral Hardy inequality on metric measure space with parameters $q < 0$ and $p \in (0, 1)$. As consequences, we obtained integral reverse Hardy inequality with parameters $q < 0$ and $p \in (0, 1)$ on homogeneous Lie groups, hyperbolic space and Cartan-Hadamard manifolds. In addition, we obtained integral reverse Hardy inequality on metric measure space with parameters $\infty < q \leq p < 0$ and a consequences we show reverse integral reverse Hardy inequality on homogeneous Lie groups. Then we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss and improved Stein-Weiss inequalities on homogeneous Lie groups with parametres $q < 0$ and $p \in (0, 1)$. Also, we obtain the reverse Hardy-Littlewood-Sobolev, Stein-Weiss type and improved Stein-Weiss type inequalities with parameters $\infty < q \leq p < 0$, which is even new in Euclidean settings. In addition, we obtain the reverse Hardy, L^p-Sobolev and L^p- Caffarelli-Kohn-Nirenberg inequalities with the radial derivative on homogeneous Lie groups.

In Chapter 5, where we investigate nonlinear PDE on groups by using our results. Firstly, we obtain Lyapunov inequalities for the fractional p-sub-Laplacian equation and systems on homogeneous Lie groups. Then, we show existence of the weak solution for the nonlinear equation with the p-sub-Laplacian on the Heisenberg and stratified groups and we show existence of the weak solution for the nonlinear equation with the fractional p-sub-Laplacian and Hardy potential on homogeneous Lie groups. Then we discussed blow-up results for heat equation with fractional p-sub-Laplacian on homogeneous Lie groups, for heat equation with fractional sub-Laplacian on stratified groups, viscoelastic equation, heat and wave Rockland equations on graded groups.

In Appendix, we considered one-dimensional functional inequalities on Euclidean case. Firstly, we obtain fractional Hardy, Poincaré type, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for the fractional order differential operators as Caputo, Riemann-Liouville and Hadamard fractional derivatives. Also, we show applications of these inequalities. In addition, we show Lyapunov and Hartman-Wintner-type inequalities for a fractional partial differential equation with Dirichlet condition, we give an application of this inequalities for the first eigenvalue and we show de La Vallée Poussin-type inequality for fractional elliptic boundary value problem.

Most of results in this dissertation were published on high peer-reviewed journals.

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